Theory
of
BLOCK DESIGN

Dr. D. K. Ghosh

Department of Statistics
Saurashtra University
RAJKOT

COMPLETE BLOCK DESIGN

If each of the blocks of a design contains each of the v treatments then such design is called complete block design .

Ex :-

1	2	3
2	3	1

here $v = 3$, $b = 2$, $k = 3$

Here $v = k$ where v = number of treatments

INCOMPLETE BLOCK DESIGN

If any one block of a design does not contain all the treatments then design becomes incomplete block design .That is $k < v$.

Example:- (i) $v = 3$, $b = 2$, $k = 2$ and

(ii) $v = 3$, $b = 3$, $k = 2$

1	2
2	3

1	2
2	3
3	3

Binary design and non Binary design

A connected design is said to be binary if the

incidence matrix N is defined as

$$N = \left(n_{ij}\right)_{v\times b} = \begin{cases} 0 & ;if \quad i^{th}\ treat\ ment \\ & \ \ doesnot\ occur\ in\ j^{th} block. \\ 1 & ;if \quad i^{th}\ treat\ ment \\ & \ \ occur\ in\ j^{th} block. \end{cases}$$

otherwise non binary design

Randomized Block Design

A block design is said to be randomized block design if v treatments are arranged in b block such that each block contains each treatments once and each treatment is replicated in r (=b) blocks

Example :- Randomized block design with v = 4 and b = 3

1	2	3	4
2	4	1	3
3	4	1	2

; v = 4 ,b = 3, r= 3, k= 4

$$N = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 \end{array}$$

RBD is a complete block design .

* Incomplete block design .

Example :- v = 4, b = 6, r =3 , k = 2.

1	2
1	3
1	4
2	3
2	4
3	4

$$
N = \begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 1 & 1 & 1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 1 & 1 & 0 \\
3 & 0 & 1 & 0 & 1 & 0 & 1 \\
4 & 0 & 0 & 1 & 0 & 1 & 1
\end{array}
$$

This is a Binary block design.

Non Binary

1	2	3
1	2	4
1	1	2
1	2	4

v	b	1	2	3	4
1		1	1	2	1
2		1	1	1	1
3		1	0	0	0
4		0	1	0	1

Properties of Block Design

(1) Connectedness
(2) Balancedness and
(3) Orthogonality.

Connectedness :- A block design is said to be connected if all the elementary treatment contrasts are estimable

Theorem: A block design is said to be connectedness iff Rank(C) = v-1

Proof: Necessary: Let a block design is connected Consider a set of (v-1) linearly independent Treatment contrast ($T_i - T_j$) if $i \neq j = 1,2,3\ldots v$ .Let the contrast be denoted by $l_j t$ where j = 1,2,3……v-1 i.e. contrasts are $l_1' t, l_2' t, l_3' t, \ldots\ldots\ldots l_{v-i}' t$ where $\underline{T}$ =(t_1, t_2, t_3,…, t_v), obviously the vector $l_1, l_2, \ldots l_{v-1}$ from the basis of a vector space of dimension v-1 . Now l_j't(t =1,2,3,…,v-1) is estimable iff it belong to the column space of the matrix of the design then

$R(C) = R(C, l_j)$ (1)

Therefore it is proved that from (1), the dimension of column space of C-matrix must be same as that of vector space spanned by the vector is (j= 1,2,3…v-1)

It follows that (v-1)=Rank (C) (2)

Now, C is a matrix of order v × v and E $_{1v}$ C=0

$$\therefore R(C) \le v-1 \qquad (3)$$

From (2) and (3)

It can be proved that R (C) = v-1 (4)

Let $\underline{l}' \underline{C} \left(E_{1v} l = 0\right)$ be one the treatment contrast ,

Now it is clear that

$$\mathrm{R}(C,l) \ge R(C) = v-1 \qquad (5)$$

But (C,l) is the matrix of order $v \times (v+1)$

$$\therefore R(C) \le v$$

Also E $_{1v}$ (C,l) = 0 ($\because$ E $_{1v}$ C=0 , E $_{1v}$ l = 0)

$$\therefore R(C,l) \le v-1 = R(C) \qquad (6)$$

From (5) and (6) it follows that

R(C) = R(C,l) =v-1 .

C = R^{δ} - $NK^{-1}N'$ is called information matrix

= diag (r_1, r_2, r_3,…, r_v) - $NK^{-1}N'$

<u>Properties of C-Matrix</u> : As a matrix

1.Each row and each column sum is zero

i.e. $C_{vv} E_{v1} = 0 = E_{1v} C_{vv}$

1 It is Doubly centroid matrix

2 Diagonal elements of C- Matrix are always non negative

3 Off diagonal element of C-Matrix are negative or zero

(4) C-Matrix is expressed as :

$$C = \theta\left(I_v - \frac{1}{v} E_{vv} \right) \qquad \text{where}$$

θ is nonzero eigen value of C-Matrix with Multiply v-1,E_{vv} is a Matrix of unit . Also C matrix can be expressed as

$$C = R^{\delta} - Nk^{-1}N'$$

(5) C-Matrix is a positive semi definite matrix

Treatment i associated with block j

$Bl\ 1$	A	B	C
2	A	C	D
3	A	D	E
4	A	E	F
5	A	F	G
$Bl\ 6$	A	B	G

Treatment

	A	*B*	*C*	*D*	*E*	*F*	*G*
1	0	0	0				
2	0		0	0			
3	0			0	0		
4	0				0	0	
5	0					0	0
6	0	0					0

Theorem : In a connected design the diagonal elements of the C-Matrix are all positive .

Proof :- Since R(C) = v-1 and $\sigma^2 C$ is the dispersion Matrix of Q. C is positive semi definite as all the given roots of C-Matrix except one are positive hence none of the diagonal elements of C-Matrix can be negative. Let if possible the i^{th} diagonal element of C be zero. Consider the vector whose i^{th} element is its only non zero element equal to 1 then $\delta' C \rho = 0$

Implying that ρ is also a characteristic vector corresponding to zero . Since ρ and E_{1v} are independent and both are characteristic vector corresponds to the zero root of C. The rank of C is at most v-2 and the design is disconnected to the contrary to the hypothesis. Hence none of diagonal elements of C-Matrix are negative or zero.

Theorem: In a connected design the co-factors of all elements of C have the same positive value.

Proof: Let C = C_{ij} and let C_{ij} be the co-factor of C_{ij} Let C^{*} = C_{ij} It is well known that $CC^{*} = D_{vv}$

Since the design is connected so a non zero scalar multiple of E_{iv} is a characteristic vector corresponding to the zero root . Hence each column of C^{*} contains identical element and become C^{*} as symmetrical and the diagonal elements of C-Matrix are all positive Hence it is a positive scalar multiple of E_{v-1} so the co-factor of all elements of C are positive .

Definition 1 (Balanced):

A connected design is said to be balance if all the treatment contrast are estimated with same variances

2. A design is said to be balance if all the treatment contrast are having same precision.
3. A design is said to be balance if all the diagonal elements of C matrix are same and off diagonal elements are also another constant

4. Balance Design: A design is said to be balance design if C-Matrix is written as

$$C = \theta\left[I_v - \frac{1}{v}E_{vv}\right] \quad \text{where}$$

θ is non zero eivenroot of C-Matrix with multiplicity (v-1). Iv is an identity matrix of order v and E_{vv} is a matrix of order v and all elements are unique .

Orthogonal :-

An I BD is said to be orthogonal if Cov (Q , P) = 0, where P= $\underline{B} - N'R^{-1}\underline{T}$ and Q= $\underline{T} - N'K^{-1}\underline{B}$

An IBD is said to be orthogonal if the incidence matrix of IBD is expressed as $N = \frac{rk^1}{n}$

<u>THEOREM</u>: An IBD is said to be orthogonal iff Cov(Q,P) = 0 when $N = \frac{rk^1}{n}$

Let N be an incidence matrix of a BIBD ,

let $N = D_1 D_2' \Rightarrow N' = D_2 D_1' \quad R = D_1 D_1'$,

$$k = D_2 D_2', T = D_1 y, B = D_2 y$$

Now Cov(Q,P) = Cov $\left[T - NK^{-1}B\ ,\ B - N'R^{-1}T\right]$

= Cov $\left[(T - NK^{-1}B)\ (B - N'R^{-1}T)'\right]$

= Cov $\left[(D_1 y - D_1 D_2' K^{-1} D_2 y)\ (D_2 y - D_2 D_1' R^{-1} D_1 y)'\right]$

= Cov $\left[(D_1 - D_1 D_2' K^{-1} D_2)\ yy'\ (D_2 - D_2 D_1' R^{-1} D_1)'\right]$

$$= (D_1 - D_1 D_2' K^{-1} D_2)\,(D_2 - D_2 D_1' R^{-1} D_1)'\sigma^2$$

$$[D_1 D_2' - D_1 D_1' R^{-1} D_1 D_2' - D_1 D_2{}' K^{-1} D_2 D_2' +$$

$$D_1 D_2{}' K^{-1} D_2 D_1' R^{-1} D_1 D_2{}']\sigma^2$$

$$=[N - RR^{-1}N - NK^{-1}K + NK^{-1}N'R^{-1}N]\sigma^2$$

$$=[N - N - N + NK^{-1}N'R^{-1}N]\sigma^2$$

$$\therefore Cov\left(Q, P\right) = NK^{-1}N'R^{-1}N - N]\sigma^2$$

Necessary: N = rk'/n is given and then we have to prove that Cov (Q,P) = 0

$$\therefore Cov(Q,P) = \left[\frac{rk'}{n} K^{-1} \frac{kr'}{n} R^{-1} N\right] - N$$

$$= n^{-2} \left[rk'K^{-1}kr'RN\right] - N$$

$$= n^{-2} \left[rE_{ib}kE_{iv}N\right] - N$$

$$= n^{-2} \left[r(E_{ib}k)E_{iv}N\right] - N = n^{-2} \left[nrE_{iv}N\right] - N$$

$$= n^{-2} \left[n\,n\,N\right] - N = N - N = 0 \quad \therefore \quad Cov(Q,P) = 0$$

Necessary: N = rk'/n is given and then we have to prove that Cov (Q,P) = 0

$$\therefore Cov(Q,P) = \left[\frac{rk'}{n} K^{-1} \frac{kr'}{n} R^{-1} N\right] - N$$

$$= n^{-2}\left[rk'K^{-1}kr'RN\right] - N$$

$$= n^{-2}\left[rE_{ib}kE_{iv}N\right] - N$$

$$= n^{-2}\left[r(E_{ib}k)E_{iv}N\right] - N = n^{-2}\left[nrE_{iv}N\right] - N$$

$$= n^{-2}\left[n\,n\,N\right] - N = N - N = 0 \quad \therefore \quad Cov(Q,P) = 0$$

Sufficient:-It is given that Cov (Q,P) = 0, now we have to prove that N = rk'/n.

$$\therefore Cov(Q,P) = [NK^{-1}N'R^{-1}N - N]\sigma^2 = 0$$

$$NK^{-1}N'R^{-1}N - N = \left(R - C\right)R^{-1}N - N$$

$$= RR^{-1}N - CR^{-1}N - N \quad \begin{cases} C = R - NK^{-1}N' \\ R - C = NK^{-1}N' \end{cases}$$

$$= N - CR^{-1}N - N \qquad = -CR^{-1}N$$

Since Cov (Q,P) = 0 $\therefore CR^{-1}N = 0$

Let $R^{-1}N = A$ it follows (Assume connected) that column of A say $a_1, a_2, a_3, \ldots\ldots.. a_b$ are proportional to i (Recall that $E_{vi} = 0$) i.e.

$a_i = \alpha_i E_{vi}$ for $i=1,2,3,\ldots\ldots b$ where α_i are some scalars.This implies that $A = R^{-1} N = E_{vi}\,\alpha'$

where $\alpha' = (\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_b)$

It is now easy to show that $\alpha' = K'_n$ so it prove

that $N = \dfrac{rk'}{n}$

BALANCED INCOMPLETE BLOCK DESIGN:

Definition:-

BIBD is an incomplete block design where v treatments are arranged in b blocks having k plots in each block $(k<v)$ such that

(1) Each treatment is replicated in r blocks and

(2) A pair of treatments occurs together in λ blocks.

1	1	2	4
2	2	3	5
3	3	4	6
4	4	5	7
5	5	6	1
6	6	7	2
7	7	1	3

In this Design v=7, b =7, r=3, k=3 and λ=1

- Parameters of BIBD:

BIBD has five parameters v, b, r, k, λ .

- Parametric relation :-

(i) vr = bk (ii)λ(v-1) = r (k-1) and

(iii) $b \geq v$ (Fisher`s inequality)

- Prove that: vr = bk

Let us consider a BIBD with parameters v, b, r, k and λ.Let N be its incidence matrix . Since BIBD is a binary and hence

$$N = (n_{ij}) = \begin{cases} 1 \\ 0 \end{cases}$$

In a BIBD $\quad r_1 = r_2 = \ldots. = r_v = r$

$\therefore E_{1v} N = kE_{1b}$

$NE_{b1} = rE_{v1}$

Now, $E_{1v} N E_{b1} = (E_{1v} N) E_{b1}$

$= k E_{1b} E_{b1} = kb \qquad (1)$

$E_{1v} N E_{b1} = E_{1v}(N \; E_{b1})$

$= E_{1v} rE_{v1} = rv \qquad (2)$

From (1) and (2), $vr = bk$

Alternative proof:

One block contains k treatments and we have b such blocks, so total no. of units will be bk.

Again one treatment is replicated r times and we have such v treatments and hence total number of units will be vr, so $vr = bk$

$$\underset{\underset{N}{\downarrow}}{\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}_{7\times 7}} \underset{\underset{E_{b1}}{\downarrow}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}_{7\times 1}} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} = \underset{\underset{r}{\downarrow}}{3} \underset{\underset{E_{v1}}{\downarrow}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}_{7\times 1}}$$

$\therefore\ NE_{b1} = rE_{v1}$

1 Prove that : $\lambda(v-1) = r(k-1)$

$$NN'E_{v1} = N(E_{1v}N)'$$

$$= N(kE_{1b})' \quad \{\because E_{1v}N = kE_{1b}$$

$$= k'NE_{b1} \quad = krE_{v1} \quad (\because NE_{b1} = rE_{v1})$$

$$NN' = \begin{bmatrix} n_{11} & n_{12} & \dots & n_{1b} \\ n_{21} & n_{22} & \dots & n_{2b} \\ \dots & \dots & \dots & \dots \\ n_{v1} & n_{v2} & \dots & n_{vb} \end{bmatrix} \begin{bmatrix} n_{11} & n_{21} & \dots & n_{v1} \\ n_{12} & n_{22} & \dots & n_{v2} \\ \dots & \dots & \dots & \dots \\ n_{1b} & n_{2b} & \dots & n_{vb} \end{bmatrix}$$

$$\begin{bmatrix} n_{11}^2 + n_{12}^2 + \ldots + n_{1b}^2 & n_{11}n_{21} + n_{12}n_{22} + \ldots + n_{1b}n_{2b} & n_{11}n_{vi} + n_{12}n_{v2} + \ldots + n_{11}n_{vi} \\ n_{21}n_{11} + n_{22}n_{12} + \ldots + n_{2b}n_{1b} & n_{21}n_{21} + n_{22}^2 + \ldots + n_{2b}^2 & n_{21}n_{v1} + \ldots + n_{2b}n_{vb} \\ \ldots & \ldots & \ldots \\ n_{vi}n_{11} + n_{v2}n_{12} + \ldots + n_{vb}n_{1b} & n_{v1}n_{21} + n_{v2}n_{22} + \ldots + n_{vb}n_{2b} & n_{v1}^2 + n_{v2}^2 + \ldots + n_{vb}^2 \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{j=1}^{b} n_{1j}^2 & \sum_{j=1}^{b} n_{1j} n_{2j} & \sum_{j=1}^{b} n_{1j} n_{vj} \\ \sum_{j=1}^{b} n_{2j} n_{1j} & \sum_{j=1}^{b} n_{2j}^2 & \sum_{j=1}^{b} n_{2j} n_{vj} \\ \sum_{j=1}^{b} n_{vj} n_{1j} & \sum_{j=1}^{b} n_{vj} n_{2j} & \sum_{j=1}^{b} n_{vj}^2 \end{bmatrix}$$

In a binary design $\sum_{j=1}^{b} n_{ij}^2 = r$ and

$\sum_{j=1}^{b} n_{ij} n_{mj} = \lambda$ for all $i \neq j, m \neq i$

$$\therefore NN' = \begin{bmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \cdots & \cdots & \cdots & \cdots \\ \lambda & \lambda & & r \end{bmatrix}_{v \times v}$$

$$\Rightarrow \begin{bmatrix} r & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & r \end{bmatrix} + \begin{bmatrix} 0 & \lambda & \cdots & \lambda \\ \lambda & 0 & \cdots & \lambda \\ \cdots & \cdots & \cdots & \cdots \\ \lambda & \lambda & \cdots & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} r & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & r \end{bmatrix} + \begin{bmatrix} \lambda & \lambda & \cdots & \lambda \\ \lambda & \lambda & \cdots & \lambda \\ \cdots & \cdots & \cdots & \cdots \\ \lambda & \lambda & \cdots & \lambda \end{bmatrix} -$$

$$\begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix}$$

$\therefore NN' = rI_v + \lambda E_{vv} - \lambda I_v$

where Iv is a matrix of order v and E_{vv} is a matrix with all elements unit

$\therefore NN' = (r-\lambda)\, I_v + \lambda E_{vv}$

$NN'\, E_{v1} = [(r-\lambda)\, I_v + \lambda E_{vv}\,]\, E_{v1}$

$= (r-\lambda)\, E_{v1} + \lambda v\, E_{v1}$

$= [(r-\lambda) + \lambda v\,]\, E_{v1} \qquad (2)$

compairing (1) and (2)

$[(r-\lambda) + \lambda v\,]\, E_{v1} = kr\, E_{v1}$

$r + \lambda v - \lambda = kr$

$r + \lambda(v-1) = kr$

$-r + kr = \lambda(v-1)$

$1 \qquad \lambda(v-1) = r\ (k-1)$

*FISHER`S INEQUALITY *

PROVE THAT : b $\geq$ v

We know that $\therefore NN' = \begin{bmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \cdots & \cdots & \cdots & \cdots \\ \lambda & \lambda & \cdots & r \end{bmatrix}_{v \times v}$

In a matrix if all off diagonal elements are zero then |M| = product of all diagonal elements.

Adding all columns in first column we get

$$NN' = \begin{bmatrix} r+\lambda+\lambda+.....+\lambda & \lambda & \cdots & \lambda \\ \lambda+r+\lambda+...+\lambda & r & \cdots & \lambda \\ \cdots & \cdots & \cdots & \cdots \\ \lambda+\lambda+r+...+r & \lambda & \cdots & r \end{bmatrix}_{v\times v}$$

$$
= \begin{bmatrix} r+\lambda(v-1) & \lambda & \cdots & \lambda \\ r+\lambda(v-1) & r & \cdots & \lambda \\ \cdots & \cdots & \cdots & \cdots \\ r+\lambda(v-1) & \lambda & \cdots & r \end{bmatrix} =
$$

$$
r+\lambda(v-1) \begin{bmatrix} 1 & \lambda & \cdots & \lambda \\ 1 & r & \cdots & \lambda \\ \vdots & \vdots & \cdots & \cdots \\ 1 & \lambda & \cdots & r \end{bmatrix}
$$

$$= [r + \lambda(v-1)] \begin{bmatrix} 1 & \theta & \cdots & \theta \\ \theta & r-\lambda & \cdots & \theta \\ \vdots & r-\lambda & \cdots & \theta \\ \theta & \theta & \cdots & r-\lambda \end{bmatrix}$$

Now, $|NN'| = [r + \lambda(v-1)]\ (r-\lambda)^{v-1}$

In a BIBD r>λ.

$\therefore |NN'| \neq 0$, so NN' is non singular matrix having dimension vxv $\therefore$ Rank $(NN') = v$

Now Rank $(NN') \leq Rank(N)$

But here N is a matrix of vxb

$\therefore$ Rank (N) =min (v,b)

If Rank (N) = v then Rank (NN') $\leq$ v

This shows b $\geq$ v

1 BOSE INEQUALITY *

THEORAM: For any BIBD $b \leq v + r - k$.

Proof: Let us consider a BIBD with parameters v, b, r, k and λ. Then we know that $vr = bk$

We also know that in a BIBD, $k \leq v$, i.e.,

$v - k \geq 0$ similarly $r \geq k$, i.e., $(r - k) \geq 0$

$$\Rightarrow (v-k)(r-k) \geq 0$$

$$vr - kr - vk + k^2 \geq 0 \quad \{\because vr = bk$$

$$\therefore bk - kr - vk + k^2 \geq 0$$

$$k(b - r - v + k) \geq 0$$

$k \neq 0$ so $(b - r - v + k) \geq 0$

$\therefore b \geq v + r - k$

Theorem: Show that a BIBD is connected

if $R(C) = v-1$

Proof: Consider a BIBD with parameters

v, b, r, k and λ. Now the information matrix C

for a BIBD is given by $C = r I_v - NK^{-1}N'$

N is the incidence matrix of BIBD, N' is the

transpose of N.

We know $C = rI_v - \frac{NN'}{k}$ because $r_1 = r_2 \ldots = r_v$

$$= rI_v - \frac{[(r-\lambda)I_v + \lambda E_{vv}]}{k}$$

$$= \left[r - \frac{(r-\lambda)}{k}\right] I_v - \frac{\lambda}{k} E_{vv}$$

$$= \left[\frac{rk - r + \lambda}{k}\right] I_v - \frac{\lambda}{k} E_{vv}$$

$$= \left[\frac{r(k-1)+\lambda}{k} \right] I_v - \frac{\lambda}{k} E_{vv}$$

$$= \left[\frac{\lambda\ (v-1)+\lambda}{k} \right] I_v - \frac{\lambda}{k} E_{vv}$$

$$= \frac{\lambda v}{k} I_v - \frac{\lambda}{k} E_{vv} \qquad = \frac{\lambda}{k} [v I_v - E_{vv}]$$

$$= \frac{\lambda v}{k} \left[I_v - \frac{E_{vv}}{v} \right] \qquad (1)$$

$$\text{Now}\left[I_v - \frac{E_{vv}}{v}\right]^2 = \left[I_v - \frac{E_{vv}}{v}\right]\left[I_v - \frac{E_{vv}}{v}\right]$$

$$= I_v - \frac{E_{vv}}{v} - \frac{E_{vv}}{v} + \frac{E_{vv}}{v}\frac{E_{vv}}{v}$$

$$= I_v - \frac{2}{v}E_{vv} + \frac{1}{v^2}v\ E_{vv}$$

$$= I_v - \frac{2}{v}E_{vv} + \frac{E_{vv}}{v} \qquad = I_v - \frac{E_{vv}}{v}$$

$$\therefore \left[I_v - \frac{E_{vv}}{v} \right]^2 = I_v - \frac{E_{vv}}{v}$$

This shows that $\left(I_v - \frac{E_{vv}}{v} \right)$ an idempotent

matrix .

Rank of any idempotent matrix = trace of a

matrix = sum of the diagonal elements

$$C = \frac{\lambda v}{k} \left[I_v - \frac{E_{vv}}{v} \right]$$

$$Rank\, C = Rank\left[I_v - \frac{E_{vv}}{v} \right]$$

$$= R(I_v) - \frac{1}{v} R\left(E_{vv}\right) = v - \frac{1}{v} v$$

Rank (C) = v-1

Remarks: BIBD is balanced if

$$C = \frac{\lambda v}{k}\left[I_v - \frac{1}{v} E_{vv} \right]$$

$$= \theta\left[I_v - \frac{E_{vv}}{v}\right] \qquad (2)$$

Where θ is eigen value of C matrix of design d with multiplicity (v-1). Here C-Matrix is singular matrix and hence one eiven value is zero and remaining (v-1) eigen value are $\frac{\lambda v}{k}$.

(Symmetrical Balanced Incomplete Block Design

A BIBD with parameters v, b, r, k, λ is called

SBIBD if v = b.The Incidence matrix of SBIBD

is always square matrix (N=N').Incidence matrix

$N_{vxb} = N_{v\,xv}$ and hence it is a square matrix.

Theorem: For any symmetrical BIBD, $(r-\lambda)$ must

be a perfect square for even v.

Proof: Let us consider a BIBD with parameters

v, b, r, k, λ. Let N be its incidence matrix and

N' is its transpose. We know that

$$|NN'| = [r+\lambda(v-1)]\,(r-\lambda)^{v-1}$$

$$= [r+r(k-1)]\,(r-\lambda)^{v-1} \qquad \{\ \lambda\,(v-1) = r\,(k-1)$$

$$= (r+rk-r)\,(r-\lambda)^{v-1} = (rk)\,(r-\lambda)^{v-1} = (rr)\,(r-\lambda)^{v-1}$$

$$|NN'| = (r^2)\,(r-\lambda)^{v-1}$$

$$|N|\,|N'| = r^2\,(r-\lambda)^{v-1}$$

$$|N|\,|N| = r^2\,(r-\lambda)^{v-1}\ \{\ N = N' \text{ for symmetrical}$$

$$|N|^2 = r^2\,(r-\lambda)^{v-1}$$

$$\therefore |N| = r\,(r-\lambda)^{\frac{v-1}{2}}$$

This shows that for any even values of v, $(r - \lambda)$

must be a perfect square.

Resolvable BIBD

A BIBD is said to be resolvable, if b blocks are arranged in r groups such that each group contain one and only one treatment. Each group will contain b/r blocks. Any two treatments common between two blocks of the same group are constant while any two treatments common between two block of the different groups are another constant.

Example: 1. $v = 4, b = 6, r = 3, k = 2, \lambda = 1.$

Design plan

1	2
1	3
1	4
2	3
2	4
3	4

Here $b = 6, r = 3, \therefore b/r = 6/3 = 2$ blocks

$$\begin{array}{cccccccl} 1 & 2 & 1 & 3 & 1 & 4 & \rightarrow Block \\ 3 & 4 & 2 & 4 & 2 & 3 & \rightarrow Block \end{array}$$

$v = 4 \qquad r = 3 \qquad k = 2$

α - Resolvable BIBD

A resolvable BIBD is said to α- Resolvable BIBD if each group contains each treatment α time

Example-1 is a 1-Resolvable BIBD also.

Affine Resolvable BIBD:

A Resolvable BIBD is said to be Affine Resolvable

BIBD if number of treatment common between

any two blocks of same group is constant similarly

any two block of different group are another constant

α - Affine Resolvable BIBD

An Affine resolvable BIBD is said to α -Affine Resolvable BIBD if in each group each treatment occur α times.

Show that: A design with parameters $v = 4$, $b = 6$, $r = 3$, $k = 2$, $\lambda = 1$ is Balanced , connected or orthogonal

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}_{4\times 6} \qquad N' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}_{6\times 4}$$

now $\quad NN' = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}_{4\times 4}$

C = diag(r) –$NK^{-1}N$

$$= diag.(r) - \frac{NN'}{k}$$

$\{\therefore\ k_1 = k_2 = \ldots = k_v\ )$

$$= \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - \frac{NN'}{k}$$

$$= \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 3/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 3/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 3/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 3/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 3/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 3/2 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 3/4 & -1/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 3/4 \end{bmatrix} \qquad (1)$$

Since all the diagonal elements are constant

and again all the off diagonal elements are

another constant. So design is Balance

$$C = \theta \left[I_v - \tfrac{1}{v} E_{vv} \right]$$

$$= \theta [\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \Big/ 4 \;]$$

$$=\theta\begin{bmatrix} 3/4 & -1/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 3/4 \end{bmatrix} \qquad (2)$$

From (1) and (2) we get $\theta = 2$

$\therefore$ Eigen value = 2 with multiply (v-1) = 3,

so design is Balance.

$\Rightarrow \qquad E_{1v}C = 0 \quad \therefore |C| = 0$

$C\,E_{1v} = 0$

C is a singular matrix so Rank (C)=v-1=3

$\therefore$ design is connected .

Here r = 3 k = 2

r = (3 3 3 3)' k = (2 2 2 2 2 2)'

$$rk' = (3.3.3.3)_{1\times 4} \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}_{6\times 1}$$

$$= \begin{pmatrix} 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 & 6 & 6 \end{pmatrix}$$

rk'/n n=vr = 4*3 = 12

$$= \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}$$

$$\therefore N \neq \frac{rk'}{n}$$

$\therefore$ Design is not an orthogonal .

All the Incomplete Block design are

non-orthogonal.

Consider a Randomized block Design with

4 treatments and 3 blocks.

1	2	3	4
2	3	4	1
2	1	3	4

now

	1	2	3
1	1	1	1
2	1	1	1
3	1	1	1
4	1	1	1

so Incidence matrix of Randomized block design is $N = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$r = (3\ 3\ 3\ 3\ )'$ $\quad$ $k = (\ 4\ 4\ 4\ )'$

$$rk' = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}_{4\times 1} \begin{pmatrix} 444 \end{pmatrix}_{1\times 3} = \begin{bmatrix} 12 & 12 & 12 \\ 12 & 12 & 12 \\ 12 & 12 & 12 \\ 12 & 12 & 12 \end{bmatrix}$$

now $\quad$ n = vr $\quad$ = 4x3 = 12.

$$\therefore \frac{rk'}{n} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = N$$

$\therefore$ this design is orthogonal .

Conclusion: All the RBD are orthogonal Design.

Consider a Latin Square Design

v = 3, b = 3, r = 3, k = 3.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Incidence matrix $N = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Now $r = (3.3.3)'$ $\quad k' = (3.3.3)$

$$rk' = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} \begin{pmatrix} 3 & 3 & 3 \end{pmatrix} = \begin{bmatrix} 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \end{bmatrix}$$

Now n= vr = 3x3 = 9

$$\therefore \frac{rk'}{n} = \begin{bmatrix} 9 & 9 & 9 \\ 9 & 9 & 9 \\ 9 & 9 & 9 \end{bmatrix} \Big/ 9$$

$$\frac{rk'}{n} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = N$$

$$N = \frac{rk'}{n}$$

$\therefore$ The design is orthogonal .

Remarks: L.S.D. is an orthogonal design.

All the Complete Block Designs are

Orthogonal Designs. All the Incomplete

Block Designs are Non orthogonal Designs.

Analysis of Intrablock BIB design.

From the analysis of one way block design

we know that the reduced normal equation

for estimating treatment effect τ is given by

$\underline{Q} = C\underline{\tau}$ where $\underline{Q} = \underline{T} - NK^{-1}B$

$C = \text{diag}(r) - NK^{-1}N'$,

where N is the incidence matrix of block design

T is vector of treatment total and B is vector

of block total.In case of BIBD we have

parameters v, b, r, k, λ

$r_1 = r_2 = \ldots. = r_v$, $k_1 = k_2 = \ldots\ldots.= k_b$

$$C = \text{diag}(r, r, \ldots.., r) - NK^{-1}N'$$

$$
= \begin{bmatrix} r & 0 & 0 & \cdots & 0 \\ 0 & r & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & r \end{bmatrix} - \frac{NN'}{k}
$$

$$
\therefore \mathrm{C} = \mathrm{rI_v} - \frac{\mathrm{NN'}}{\mathrm{k}} \ \{\mathrm{NN'} = (\mathrm{r}\text{-}\lambda)\ \mathrm{Iv} + \lambda \mathrm{E_{vv}}
$$

$$
= r - [(r-\lambda)\, I_v + \lambda E_{vv}] / k
$$

$$
= \left[r - \frac{(r-\lambda)}{k} \right] I_v - \frac{\lambda E_{vv}}{k}
$$

$$= \left[\frac{(rk - r + \lambda)}{k} \right] I_v - \frac{\lambda E_{vv}}{k}$$

$$= \left[\frac{r(k-1) + \lambda)}{k} \right] I_v - \frac{\lambda E_{vv}}{k}$$

$$= \left[\frac{\lambda(v-1) + \lambda)}{k} \right] I_v - \frac{\lambda E_{vv}}{k}$$

$$\therefore \mathrm{C} = \frac{\lambda \mathrm{v}}{\mathrm{k}} \mathrm{I}_{\mathrm{v}} - \frac{\lambda \mathrm{E}_{\mathrm{vv}}}{\mathrm{k}}$$

now $Q = C\underline{\iota}$

$$= \left[\frac{\lambda v}{k} I_v - \frac{\lambda E_{vv}}{k} \right] \underline{\tau}$$

$$= \frac{\lambda v \underline{\tau}}{k} I_v - \frac{\lambda E_{vv} \underline{\tau}}{k}$$ since $E_{vv} \tau = 0$

$$\text{so } Q = \frac{\lambda v \tau}{k}$$

$$\therefore \underline{\tau} = \frac{k}{\lambda v} \underline{Q}$$

Now S.S. due to treatment =

$$\tau'\underline{Q} = \frac{k}{\lambda v}\underline{Q}'Q = \frac{k}{\lambda v}\sum_{i=1}^{v} Q_i^{2}$$

$$Q = T - NK^{-1}B = T - \frac{NB}{k}$$

where Q is the vector of adjusted treatment total

Variance of Treatment contrasts:

For a BIBD $\underline{\hat{\tau}} = \dfrac{k}{\lambda v} Q \Rightarrow \hat{\tau}_i = \dfrac{k}{\lambda v} Q$

$$\hat{\tau}_1 = \frac{k}{\lambda\nu} Q_1,\ \hat{\tau}_2 = \frac{k}{\lambda\nu} Q_2, \ldots, \hat{\tau}_j = \frac{k}{\lambda\nu} Q_j$$

now $\quad V(\hat{\tau}_i - \hat{\tau}_j) = V\left(\frac{k}{\lambda\nu} Q_i - \frac{k}{\lambda\nu} Q_j\right)$

$$= \left(\frac{k}{\lambda\nu} + \frac{k}{\lambda\nu}\right)\sigma^2 = \frac{2k}{\lambda\nu}\sigma^2$$

Efficiency factor of BIBD

Let E be the efficiency factor of BIBD

$$\mathrm{E} = \frac{v(\hat{\tau}_i - \hat{\tau}_j)\ \ RBD}{v(\hat{\tau}_i - \hat{\tau}_j)\ \ BIBD}$$

$$= \left(\frac{1}{r} + \frac{1}{r}\right)\sigma^2 \Big/ \frac{2k}{\lambda v}\sigma^2 = \frac{2}{r} \Big/ \frac{2k}{\lambda v} = \frac{\lambda v}{rk}$$

.Example: v = 4, b = 6, r = 3, k = 2, λ = 1

$$E = \frac{rv}{rk} = \frac{4}{3 \times 2} = \frac{2}{3} < 1$$

Construction of BIBD

METHOD :1

BIBD with a series v, b = vC_2, r = v -1, k = 2, λ =1.

Step 1: Take v treatments write down all possible combination of v treatments taking two treatments together .

Step 2: Here there will be vC_2 combinations. Two treatments are taken together and is kept in one block so these vC_2 treatment combinations are kept in vC_2 books. Each block contains 2 treatments and each treatment is replicated r times and finally a pair of treatment occurs together in λ block

Example: $v = 7$, $b = {}_7C_2 = 21$, $r = 7-1 = 6$, $k = 2$, $\lambda = 1$.

Treatments are $t_1, t_2, t_3, t_4, t_5, t_6, t_7$

Combination :

t_1t_2	t_2t_3	t_3t_4	t_4t_5	t_5t_6	t_6t_7
t_1t_3	t_2t_4	t_3t_5	t_4t_6	t_5t_7	
t_1t_4	t_2t_5	t_3t_6	t_4t_7		
t_1t_5	t_2t_6	t_3t_7			
t_1t_6	t_2t_7				
t_1t_7					

Here each treatments are replicated 6 times

$\therefore$ $r = 6$, and $k = 2$

Each pair of treatment occurs only one time.

$\therefore$ $v = 7$, $b = 21$, $r = 6$, $k = 2$, $\lambda = 1$.

METHOD 2:

v, b = ${}^{v}C_k$, $r = \binom{v-1}{k-1}, \quad \lambda = \binom{v-2}{k-2}$ for any k.

Step 1: Take v treatments write down all possible combination of v treatment taking k treatments together.

Step 2: Since there will be vC_k combinations when k treatments are taken together. Keep these combinations in vC_k blocks such that each blocks will contain k treatments. In these way each treatment is replicated $\binom{v-1}{k-1}$ times and a pair of treatment occur together in λ block .

Example : v =7,   k = 4 b = ${}_7C_4$ = 35, r = ${}_6C_3$= 20,

λ = ${}_5C_2$ = 10.

Treatments are $t_1, t_2, t_3, t_4, t_5, t_6, t_7$

$t_1t_2t_3t_4$	$t_1t_3t_4t_5$	$t_2t_3t_4t_5$	$t_3t_4t_5t_6$
$t_1t_2t_3t_5$	$t_1t_3t_4t_6$	$t_2t_3t_4t_6$	$t_3t_4t_5t_7$
$t_1t_2t_3t_6$	$t_1t_3t_4t_7$	$t_2t_3t_4t_7$	$t_3t_4t_6t_7$
$t_1t_2t_3t_7$	$t_1t_3t_5t_6$	$t_2t_3t_5t_6$	$t_3t_5t_6t_7$
$t_1t_2t_4t_5$	$t_1t_3t_5t_7$	$t_2t_3t_5t_7$	$t_4t_5t_6t_7$
$t_1t_2t_4t_6$	$t_1t_3t_6t_7$	$t_2t_3t_6t_7$	
$t_1t_2t_4t_7$	$t_1t_4t_5t_6$	$t_2t_4t_5t_6$	
$t_1t_2t_5t_6$	$t_1t_4t_5t_7$	$t_2t_4t_5t_7$	
$t_1t_2t_5t_7$	$t_1t_4t_6t_7$	$t_2t_4t_6t_7$	
$t_1t_2t_6t_7$	$t_1t_5t_6t_7$	$t_2t_5t_6t_7$	

Here each pair occur 10 times $\therefore \lambda = 10$

Method 3**:** USING LATIN SQUARE DESIGN

Step 1: Consider a Latin square design of size S which is having S rows and S column .

Step 2: Delete a column from Latin square design.

Step 3 : Consider row as a block of BIBD and Latin letters as a treatments. This way we get a BIBD with parameters :

$v = S$, $b = S$, $r = S-1$, $k = S-1$, and $\lambda = S-2$.

Example 1:- Construct a LSD of size 5

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{array} \Rightarrow \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 \\ 5 & 1 & 2 & 3 \end{array}$$

Here v = 5, b=5, r=5-1=4, k=5-1=4, & λ=5-2=3.

Example 2:- Size - 4

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{matrix} \Rightarrow \begin{matrix} 2 & 3 & 4 \\ 3 & 4 & 1 \\ 4 & 1 & 2 \\ 1 & 2 & 3 \end{matrix}$$

Here $v = 4$, $b = 4$, $r = 3$, $k = 3$, & $\lambda = 2$.

Method – 4. Using Hadamard Matrix :-

A matrix H_n of order n is said to be Hadamard Matrix if $H_n \; H_n' = nI_n = H_n' \, H_n$

First Hadamard Matrix is $H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Now $H_2 \; H_2'$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} = 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2I_n$$

Now $H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

Method 4: Using Hadamarad Matrix {-1 as +1, & 1 as 0}

Step1: Consider a Hadamard Matrix of order (size) n

Step 2: Delete first row and first column of Hadamard Matrix H_n .

Step 3: change –1 as +1 & 1 as 0.

Step 4: Consider the remaining row and column of Hadamard Matrix as the Incidence Matrix N.

Step 5: This Incidence matrix is the Incidence matrix of a BIBD with parameters.

v = n-1, b = n-1, r = n/2, k = n/2, & λ =n/4

Example: $H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \Rightarrow \begin{matrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{matrix}$

-1 as +1 & 1 as 0.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = N \ \textit{which is Incidence}$$

$$\textit{Matrix of a BIBD} \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$$

Parameters of this BIBD are $v = 3$ $(n-1) = (4-1) = 3$.$b = 3$, $r = n/2 = 4/2 = 2$, $k = n/2 = 4/2 = 2$, $\lambda = n/4 = 4/4 = 1$.

Example :-

$$H_8 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

-1as 1 & 1 as 0 .

$$\begin{bmatrix} -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

= N (Incidence matrix)

$$=\begin{bmatrix} 1 & 3 & 5 & 7 \\ 2 & 3 & 6 & 7 \\ 1 & 2 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 6 \\ 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 7 \end{bmatrix}$$

Here : treatment v = 7 { 1,2,3,4,5,6,7}

b = n-1 = 8-1 = 7, r = n/2 = 8/2 = 4,

k = n/2 = 8/2 = 4, λ = n/4 = 8/4 =2.

METHOD 5:

Using Hadamard Matrix {-1as 0 and 1 as1}

Step: All the step are same as Method 4 only -1 as 0 and 1 as 1 .

Parameters: v=b=n-1, r = k = $\frac{n}{4}-1$,λ = $\frac{n}{4}-1$.

Example :- delete first row and first column.

$$H_8 \Rightarrow \begin{bmatrix} -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

These two Methods Gives always Symmetrical BIBD.

-1 as 0 and 1as 1

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} = N\ Incidence \quad matrix$$

$$\begin{bmatrix} 2 & 4 & 6 \\ 1 & 4 & 5 \\ 3 & 4 & 7 \\ 1 & 2 & 3 \\ 2 & 5 & 7 \\ 1 & 6 & 7 \\ 3 & 5 & 6 \end{bmatrix}$$

Here,

$$v = 7$$

$$b = 7$$

$$r = \frac{n}{2} - 1 = 4 - 1 = 3$$

$$k = \frac{n}{2} - 1 = 4 - 1 = 3$$

$$\lambda = \frac{n}{4} - 1 = 2 - 1 = 1$$

METHOD: 6. BIBD with series ,

$v = 4\lambda + 3$, $b = 4\lambda + 3$, $r = 2\lambda + 1 = k$ and λ, where $4\lambda + 3$ is a prime number .

Step 1:- Let $4\lambda + 3$ is a prime number for any $\lambda > 0$. First of all find out the primitive elements of GF($4\lambda + 3$) {GF = Galois Field }. The elements of GF($4\lambda + 3$) are $0,1,2,3,\ldots,4\lambda + 3-1$ ($= 4\lambda + 2$) .

Step 2 :- Find the primitive element α for GF($4\lambda + 3$). That is, if $\alpha^{(4\lambda + 3)} - 1=1$ with reduced mode ($4\lambda + 3$) then α is primitive element .Next write all the element as the power of primitive elements. Consider either even power of primitive element α or odd power of primitive element with reduce mode $4\lambda + 3$ and keep them in block .

Denote this block as a key block. Develop this key block with reduced mod $v = 4\lambda + 3$.

This way we get BIBD with parameter

$v = 4\lambda + 3 = b$, $r = 2\lambda + 1 = k$, λ

Example: $\lambda=1$, $v= 4(1)+3=7$, $b=7$, $r=2(1)+1=3$, $k=3$.

$\therefore$ v (= 7) is a prime number.

Element of GF(7) are 0,1,2, 3, 4, 5, 6.

2^0	2^1	2^2	2^3	2^4	2^5	2^6
1	2	4	$\underline{\underline{1}}$			

$\therefore 2^6 \neq 1$ so 2 is not the primitive element of GF(7).

3^0	3^1	3^2	3^3	3^4	3^5	3^6
1	3	2	27	(6×3)	(4×3)	(5×3)
			6	18	12	15
				4	5	1

Here $3^6 = 1$ with reduced mod 7, so 3 is primitive element of GF(7).

Key by block	3^1	3^3	3^5
$b1$	3	6	5
$b2$	4	7	6
$b3$	5	1	7
$b4$	6	2	1
$b5$	7	3	2
$b6$	1	4	3
$b7$	2	5	4

Even no.

3^2	3^4	3^6
2	4	1
3	5	2
4	6	3
5	7	4
6	1	5
7	2	6
1	3	7

Here , $v=7$, $b=7$, $r=3$, $k=3$, $\lambda=1$.

Example: λ=2, gives v = 4(2) + 3 = 11=b.,
r=2(2)+1 =5, k=5,

Here v=11 is a prime number and elements of GF(11) are 0,1, 2, 3, 4, 5, 6, 7, 8, 9,10

2^0	2^1	2^2	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
1	2	4	8	16		20	18	14		12
				5	10	9	7	3	6	1

$\therefore\ 2^{11-1} = 2^{10} = 1$, so 2 is primitive element of GF(11)

	2^1	2^3	2^5	2^7	2^9
1	2	8	10	7	6
2	3	9	11	8	7
3	4	10	1	9	8
4	5	11	2	10	9
5	6	1	3	11	10
6	7	2	4	1	11
7	8	3	5	2	1
8	9	4	6	3	2
9	10	5	7	4	3
10	11	6	8	5	4
11	1	7	9	6	5

here , $v=11$, $b=11$, $r=5$, $k=5$, $\lambda=2$.

METHOD: 7. Complementary Designs:

Step 1:- Complementary Design can be obtain from the existing BIBD with parameters v, b, r, k and λ.Take a block and see, which treatments it contain, next write down those treatments which are absent in that block and keep them in another block. These way write down treatments from all blocks . This will give a BIBD with parameter $v = v_1$, $b = b_1$, $r = b – r_1$, $k= v - k_1$,

$\lambda = b_1 - {}_2r_1 + \lambda_1$

METHOD: 8. Using Block Section

Step 1:- Consider a BIBD with parameters v_1, b_1, r_1, k_1, and λ_1

Step 2: Delete any one block from this BIBD

Step 3: So the remaining blocks are now b-1

Step 4: Take one block and see which treatment are present in this block. Now select those treatments from that block which are absent in deleted block and then keep these treatments in another block .

Step 5- Continue step 4 for remaining blocks

Step 6: Since k_1 treatments are deleted from v_1

so the treatments for new design will be (v_1-k_1).
Next code it as 1,2,…….,v_1 - k

Step 7: Each block will contain (k-λ) treatment so the new BIBD exist with parameter $v = v_1 - k_1$
$b = b_1 - 1$, $k = k_1 - \lambda$, $r = r_1$, $\lambda = \lambda_1$

Example :- $v = b = 11$, $r = k = 5$, $\lambda = 2$.

2	8	10	7	6
3	9	11	8	7
4	10	1	9	8
5	11	2	10	9
6	1	3	11	10
7	2	4	1	11
8	3	5	2	1
9	4	6	3	2
10	5	7	4	3
11	6	8	5	4
1	7	9	6	5

$\Rightarrow$

2	8	10
3	11	8
4	10	8
11	2	10
3	11	10
2	4	11
8	3	2
4	3	2
10	4	3
11	8	4

$2 \rightarrow 1 \quad 8 \rightarrow 4$

$3 \rightarrow 2 \quad 10 \rightarrow 5$

$4 \rightarrow 3 \quad 11 \rightarrow 6$

treatment = 6

here, treatments = $v - k = 11 - 5 = 6$, block $b' = \{b-1\} = 10$, $r' = r = 5$, $k' = k - \lambda = 5-2 = 3$, $\lambda' = \lambda = 2$. The resulting BIBD is

1	4	5	$\rightarrow$	b_1
2	6	4		b_2
3	5	4		b_3
6	1	5		b_4
2	6	5		b_5
1	2	6		b_6
4	2	1		b_7
3	2	1		b_8
5	3	2		b_9
6	4	3		b_{10}

METHOD: 9. Block Intersection

Step 1: Consider a BIBD with parameters v_1, b_1, r_1, k_1 and λ_1.

Step 2: Delete any one block from this BIBD

Step 3: So the remaining blocks are b_1-1

Step 4: Take one block and see which treatments are present in the deleted block. Now select those treatments from this block which are present in deleted block and then keep these treatments in another block .

Step 5: Continue step 4 for remaining block .

Step 6: Since k_1 treatments remain, so for new

BIBD $v = k_1$. Now recode the treatment as 1, 2, 3,.. k_1

Step 7: Each block will contain λ_1 treatment

Step 8: So the new BIBD exist with parameters $v = k_1$,$b = b_1-1$, $r = r_1- 1$, $k = \lambda$, $\lambda = \lambda_1-1$.

$$
\begin{array}{ccccc}
2 & 8 & 10 & 7 & 6 \\
3 & 9 & 11 & 8 & 7 \\
4 & 10 & 1 & 9 & 8 \\
5 & 11 & 2 & 10 & 9 \\
6 & 1 & 3 & 11 & 10 \\
7 & 2 & 4 & 1 & 11 \\
8 & 3 & 5 & 2 & 1 \\
9 & 4 & 6 & 3 & 2 \\
10 & 5 & 7 & 4 & 3 \\
11 & 6 & 8 & 5 & 4 \\
1 & 7 & 9 & 6 & 5
\end{array}
\Rightarrow
\begin{array}{cc||}
6 & 7 \\
7 & 9 \\
1 & 9 \\
5 & 9 \\
1 & 6 \\
1 & 7 \\
1 & 5 \\
6 & 9 \\
5 & 7 \\
5 & 6
\end{array}
\quad
\begin{array}{ccc}
1 & - & 1 \\
5 & - & 2 \\
6 & - & 3 \\
7 & - & 4 \\
9 & - & 5
\end{array}
\quad
\begin{array}{cc}
3 & 4 \\
4 & 5 \\
1 & 5 \\
2 & 5 \\
1 & 3 \\
1 & 4 \\
1 & 2 \\
3 & 5 \\
2 & 4 \\
2 & 3
\end{array}
$$

Here $v = k_1 = 5$, $r = r_1 - 1 = 4$, $b = b_1 - 1 = 10$,
$k = \lambda_1 = 2$, $\lambda = \lambda_1 - 1 = 2 - 1 = 1$

METHOD:10 Projective Geometry (PG(N, s)).

Bose(1936) uses the projective geometry to construct the BIBD . Further with the help of Galois Field GF(p^n), one can construct a finite projective geometry of N dimension in the following manner:

Let $x_0, x_1, x_2, \ldots, x_N$ be the ordered set of (N+1) elements where x_i, $i = 1,2\ldots N \in$ GF(p^n) (1)

and are not simultaneously zero, will be called a point of PG(N, p^n) where s= p^n equation (1) is also called ordinate of points.

Next corresponding to x =x_1, x_2, x_3,…. x_N, we may have another set y_0, y_1, … y_N. Now it can be easily solved that no. of points in PG(N, p^n) is exactly

$$s^N + s^{N-1} + s^{N-2} + \ldots + 1 = \frac{s^{N+1} - 1}{s - 1} \quad (2)$$

All the points which satisfy the set of (N-m) homogeneous linear equation given by :

$$\left.\begin{aligned}
&a_{10}x_0 + a_{11}x_1 + \ldots\ldots\ldots\ldots\ldots + a_{1N}x_N = 0\\
&a_{20}x_0 + a_{21}x_1 + \ldots\ldots\ldots\ldots\ldots + a_{2N}x_N = 0\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&a_{(N-M)0}x_0 + a_{(N-M)1}x_1 + \ldots + a_{(N-M)N}x_N = 0
\end{aligned}\right\} (3)$$

may be set for (N-m) dimensional sub space & briefly m-flats in PG(N, p^n). Equation (3) may be said to represent this flats. However any other set of (N-m) independent equation which can be obtained by linear combination of the equation (3) will have the same set of solution and will represent the same m-flats.We call one flats a line and 2 flats a plan, the number of m flats in PG(N, p^m) is given by

$$\phi(N,m,s)=\frac{\left(s^{N+1}-1\right)\left(s^{N}-1\right)\ldots\left(s^{N-m+1}-1\right)}{\left(s^{m+1}-1\right)\left(s^{m}-1\right)\ldots\left(s-1\right)} \qquad (4)$$

To every point PG(N, p^n),

let they correspond a variety to every m-flat.

Let the correspond to a block containing of these variety whose correspond point occur in the m-flat, Points = ϕ(N, m, s)

Parameters of BIBD	points of PG
v= no. of treatment	$\frac{\left(s^{N+1}-1\right)}{\left(s-1\right)}$ or $\phi(N,o,s)$
b= no. of blocks.	$\phi(N,m,s)$
r = no. of times each treatment is repeated	$\phi((N-1),m-1,s)$
k= block size.	$\phi(N,o,s)=\frac{\left(s^{m+1}-1\right)}{\left(s-1\right)}$
λ= pair of treatment occur together	$\phi(N-2,m-2,s)$

Step1: Consider the parameters of a BIBD.

Step 2: Using the points of PG(N,s) and PG(m, s), find out the value of N,m, s. Further find $v = \dfrac{s^{N+1}-1}{s-1}$ and $k = \dfrac{s^{m+1}-1}{s-1}$ for particular value of s.

Step 3: Consider (N+1) co-ordinate in PG(N,s.).

Step 4: Develop s^{N+1} possible combinations using s={1,2,…,s-1}

Step 5: Code each possible combination as a treatment

Step 6: Now consider the Homogeneous equations

$a_0 x_0 + a_1 x_1 + \ldots + a_N x_N$ (1) where $a_i \in GF(s)$.

$x_0, x_1, x_2, \ldots x_N$ are N + 1 co-ordinate in points of GF(s).

Step 7: If any combination satisfy (1) then keep such combination in a block.This way we can get all the b blocks.

Hence we get a BIBD with parameters

v = $\dfrac{s^{N+1}-1}{s-1}$, b = $\phi(N, m, s)$.

r = $\phi(N-1, m-1, s)$, k = $\phi(N, o, s) = \dfrac{\left(s^{m+1}-1\right)}{(s-1)}$

λ = $\phi(N-2, m-2, s)$

This method gives symmetric BIBD.

Example:

v = 15, b = 15 , r = 7, k = 7, λ = 3.

$$v = \frac{s^{N+1}-1}{s-1} \Rightarrow 15 = \frac{s^{N+1}-1}{s-1},$$

Consider s = 2, so $15 = \dfrac{2^{N+1}-1}{2-1} \Rightarrow 15 = 2^{N+1}-1$

$$\Rightarrow 16 = 2^{N+1}-1 \Rightarrow N = 3.$$

$$\therefore k = \frac{s^{m+1}-1}{s-1} \quad or \quad 7 = \frac{2^{m+1}-1}{2-1},$$

$$7 = 2^{m+1}-1 \Rightarrow 8 = 2^{m+1}. \Rightarrow m = 2$$

our points are PG(3,2) & PG(2,2)

number of co-ordinates = N+1 = 3+1= 4

s = level of co-ordinate = 2 i.e. {0,1}

$\therefore\ 2^4$ possible combinations are to be developed

Homogeneous equations are given by

$x_i = 0\ (i = 0,1,2,3)$	$= 4$	${}_4C_1$
$x_i + x_j = 0 \quad i \neq j = 0,1,2,3$	$= 6$	${}_4C_2$
$x_i + x_j + x_k = 0 \quad i \neq j \neq k = 0,1,2,3$	$= 4$	${}_4C_3$
$x_0 + x_1 + x_2 + x_3 = 0$	$= 1$	${}_4C_4$
Total	$= 15$	block

x_0	x_1	x_2	x_3									
0	0	0	1	t_1	$x_0 = 0$	t_1	t_2	t_3	t_4	t_5	t_6	t_7
0	0	1	0	t_2	$x_1 = 0$	t_1	t_2	t_3	t_8	t_9	t_{10}	t_{11}
0	0	1	1	t_3	$x_2 = 0$	t_1	t_4	t_5	t_8	t_9	t_{12}	t_{13}
0	1	0	0	t_4	$x_3 = 0$	t_2	t_4	t_6	t_8	t_{10}	t_{12}	t_{14}
0	1	0	1	t_5	$x_0 + x_1 = 0$	t_1	t_2	t_3	t_{12}	t_{13}	t_{14}	t_{15}
0	1	1	0	t_6	$x_0 + x_2 = 0$	t_1	t_4	t_5	t_{10}	t_{11}	t_{14}	t_{15}
0	1	1	1	t_7	$x_0 + x_3 = 0$	t_2	t_4	t_6	t_9	t_{11}	t_{13}	t_{15}
1	0	0	0	t_8	$x_1 + x_2 = 0$	t_1	t_6	t_7	t_8	t_9	t_{14}	t_{15}
1	0	0	1	t_9	$x_1 + x_3 = 0$	t_2	t_5	t_7	t_8	t_{10}	t_{13}	t_{15}
1	0	1	0	t_{10}	$x_2 + x_3 = 0$	t_3	t_4	t_7	t_8	t_{11}	t_{12}	t_{15}
1	0	1	1	t_{11}	$x_0 + x_1 + x_2 = 0$	t_1	t_6	t_7	t_{10}	t_{11}	t_{12}	t_{13}
1	1	0	0	t_{12}	$x_0 + x_1 + x_3 = 0$	t_2	t_5	t_7	t_9	t_{11}	t_{12}	t_{14}
1	1	0	1	t_{13}	$x_0 + x_2 + x_3 = 0$	t_3	t_4	t_7	t_9	t_{10}	t_{13}	t_{14}
1	1	1	0	t_{14}	$x_1 + x_2 + x_3 = 0$	t_3	t_5	t_6	t_8	t_{11}	t_{13}	t_{14}
1	1	1	1	t_{15}	$x_0 + x_1 + x_2 + x_3 = 0$	t_3	t_5	t_6	t_9	t_{10}	t_{12}	t_{15}

This is BIBD with parameters v = 15, b = 15, r = 7, k=7, λ=3.

Example: Construct a BIBD using PG (N, s) where N=2, s=2, m=1.

In a BIBD v = $\dfrac{s^{N+1}-1}{s-1} = 2^{2+1} - 1 = 8 - 1 = 7$

$$k = \frac{s^{m+1}-1}{s-1} = \frac{2^{1+1}-1}{1} = 4 - 1 = 3$$

$$b = \phi(N, m, s) = \frac{\left(s^{N+1}-1\right)\left(s^{N}-1\right)\ldots\ldots\left(s^{N-m+1}-1\right)}{\left(s^{m+1}-1\right)\left(s^{m}-1\right)\ldots\ldots\left(s-1\right)}$$

$$\phi(2,1,2)=\frac{\left(2^{2+1}-1\right)\left(2^{2}-1\right)}{\left(2^{1+1}-1\right)\left(2^{1}-1\right)}=\frac{7\times 3}{3}=7$$

$$\mathrm{r}=\phi(N-1,m-1,s)=\ \phi(1\ ,0\ ,2)$$

$$=\frac{s^{1+1}-1}{s-1}=\frac{2^{2}-1}{2-1}=3 \qquad \therefore r=3$$

$$\lambda=\phi(N-2,m-2,s)=\ \phi(0\ ,-1\ ,2)\ \therefore \lambda=1$$

{ because of m = 1}

$\therefore$ Parameters of BIBD are v = 7, b = 7, r = 3, k = 3, λ = 1.

$r = 3$, $k = 3$, $\lambda = 1$.

Since $N = 2$, so the no. of treatment $(2+1)=3$. (i.e. x_0, x_1, x_2)

s= level of treatment =2, i.e. {0,1}, so possible number of total points $=2^3$ (=8) which are following.

x_0	x_1	x_2						
0	0	1	t_1	$x_0 = 0$	$\pmod 2$	t_1	t_2	t_3
0	1	0	t_2	$x_1 = 0$	$\pmod 2$	t_1	t_4	t_5
0	1	1	t_3	$x_2 = 0$	$\pmod 2$	t_2	t_4	t_6
1	0	0	t_4	$x_0 + x_1 = 0$	"	t_1	t_6	t_7
1	0	1	t_5	$x_0 + x_2 = 0$	"	t_2	t_5	t_7
1	1	0	t_6	$x_1 + x_2 = 0$	"	t_3	t_4	t_7
1	1	1	t_7	$x_0 + x_1 + x_2 = 0$	"	t_3	t_5	t_6

$x_i = 0$, $i = 0,1,2$ $= 3$

$x_i + x_j = 0$ $i \neq j = 0,1,2$ $= 3$

$x_0 + x_1 + x_2 = 0$ $= 1$

Total r $= 7$ block

This is a BIBD with parameters $v = 7$, $b = 7$, $r = 3$, $k = 3$, $\lambda = 1$.