Using Latin Square Design: (LSD)

A design is said to be LSD if s treatment are arranged in s rows and s columns such that each treatment occur s times and each treatment occur in each row and each column once and only once .

Example: $\begin{bmatrix} A & B & C \\ B & C & A \\ C & A & B \end{bmatrix}$

Orthogonal Latin Square Design: (OLSD)

If one LSD is superimposed on other LSD in such a way that each possible combination of Latin Letter occur once and only once then such two LSD is called orthogonal LSD.

Example: $\begin{bmatrix} A & B & C \\ B & C & A \\ C & A & B \end{bmatrix} \quad \begin{bmatrix} A & B & C \\ C & A & B \\ B & C & A \end{bmatrix} \Rightarrow \begin{bmatrix} AA & BB & CC \\ BC & CA & AB \\ CB & AC & BA \end{bmatrix}$

Mutually Orthogonal Latin Square Design (MOLSD):
If there are more than two OLS Design then they are said to be MOLS Design if all the latin square design are pair wise orthogonal.
Galois Field:
A field is said to be G.F. if it is closed under the operation of +, -, × and division(÷) . Also if it satisfies N = R mod D. For an example, 7 = 2 mod 5
N= number , R = Remainder, D = Divider.
The elements of GF(s) are 0,1 ,2, 3, …, s-1, where s is a prime number.
Example:- Let 7 is a prime number. The elements of GF(7) are
GF(7) = 0,1,2,3,4,5,6

These elements follows the operation of Galois field as

3+5 = 8 =1 mod 7 0-6 = -6 = 7 – 6 =1 mod 7

1+0 = 1 =1 mod 7 3x6 = 18 =4 mod 7

○ = 2 = 2 mod 7

$$\frac{5}{2} = 0 + \frac{5}{2} = \frac{7+5}{2} = \frac{12}{2} = 6 \text{ mod } 7$$

$$\frac{6}{4} = \frac{1 \times 6}{4} = \frac{8 \times 6}{4} = 12 = 5 \text{ mod } 7$$

3^1	3^2	3^3	3^4	3^5	3^6
3	9	27	18	12	15
	2	6	4	5	1

Hence 3 is the primitive element of GF(7).

Construction of MOLS

Case 1: When s is a prime number.

Case 2: When s is a complex number.

Case:1 Method for obtaining MOLS for size s, when s is a prime number

Step 1: Write the element of GF(s) from 0,1,2,……,s-1.

Step 2: Take any one element from it , say, α and then write all the elements of GF(s) as its power

Step 3: Cheek α^{s-1} , that is, if $\alpha^{s-1} = 1$ then such element, α is called primitive element of GF(s) when s is a prime number.

Step 4: Write all the elements as power of α and developed it with reduced mod s

Step 5: Keep all these elements in a row and call it as a key row .

Step 6: Again keep these elements in a column and call it a key column.

Step 7: Write all possible combination of key row and key columns with reduce mod s.

Example: 3. 3 is a prime number.The elements of GF(3) are GF(3) = 0,1,2.

2^0	2^1	2^2
1	2	4
		1

2^{3-1} =1 reduced mod 3.

So 2 is a primitive element of GF(3)

key	2^0	2^1	2^2
column	0	2	1
0	0	2	1
2	2	1	0
1	1	0	2

MOLSD

Key row	0	2	1
0	0	2	1
1	1	0	2
2	2	1	0

Possible Combination

0 0	2 2	1 1
2 1	1 0	0 2
1 2	0 1	2 0

Example: Consider a number 5. Here 5 is a prime number.
The elements of GF(5) are 0, 1, 2, 3, 4.

2^0	2^1	2^2	2^3	2^4
	2	4	8	6
			3	1

$2^{5-1} = 2^4 = 1$ reduced mod 5, so 2 is a primitive element of GF(5)

	2^0	2^1	2^2	2^3	2^4
	0	2	4	3	1
0	0	2	4	3	1
2	2	4	1	0	3
4	4	1	3	2	0
3	3	0	2	1	4
1	1	3	0	4	2

0	2	4	3	1
4	1	3	2	0
3	0	2	1	4
1	3	0	4	2
2	4	1	0	3

0	2	4	3	1
3	0	2	1	4
1	3	0	4	2
2	4	1	0	3
4	1	3	2	0

0	2	4	3	1
1	3	0	4	2
2	4	1	0	3
4	1	3	2	0
3	0	2	1	4

Example 7 : GF(7) = 0,1,2,3,4,5,6.

2^1	2^2	2^3	2^4	2^5	2^6
2	4	1			

$2^3 = 1$ and $2^{7-1} \neq 1$

$\therefore$ 2 is not a primitive element of GF(S)

3^1	3^2	3^3	3^4	3^5	3^6
3	2	6	18	12	15
			4	5	1

$3^{7-1} = 3^6 = 1$ $\therefore$ 3 is the primitive element of GF(7)

	0	3	2	6	4	5	1
0	0	3	2	6	4	5	1
3	3	6	5	2	0	1	4
2	2	5	4	1	6	0	3
6	6	2	1	5	3	4	0
4	4	0	6	3	1	2	5
1	1	4	3	0	5	6	2

0	3	2	6	4	5	1
2	5	4	1	6	0	3
6	2	1	5	3	4	0
4	0	6	3	1	2	5
5	1	0	4	2	3	6
1	4	3	0	5	6	2
3	6	5	2	0	1	4

0	3	2	6	4	5	1
6	2	1	5	3	4	0
4	0	6	3	1	2	5
5	1	0	4	2	3	6
1	4	3	0	5	6	2
3	6	5	2	0	1	4
2	5	4	1	6	0	3

0	3	2	6	4	5	1
4	0	6	3	1	2	5
5	1	0	4	2	3	6
1	4	3	0	5	6	2
3	6	5	2	0	1	4
2	5	4	1	6	0	3
6	2	1	5	3	4	0

0	3	2	6	4	5	1
1	4	3	0	5	6	2
3	6	5	2	0	1	4
2	5	4	1	6	0	3
6	2	1	5	3	4	0
4	0	6	3	1	2	5
5	1	0	4	2	3	6

0	3	2	6	4	5	1
5	1	0	4	2	3	6
1	4	3	0	5	6	2
3	6	5	2	6	1	4
2	5	4	1	6	0	3
6	2	1	5	3	4	0
4	0	6	3	1	2	5

Case 2: MOLS for size s

When s is a complex number

Let s is a complex number i.e. it is a Non prime number then the elements of GF($s = p^n$), where p is a prime number and n is a positive integer are, 0, α^0 , α^1 ,α^2 ,…,α^{p^n-2} , where α is the primitive element of GF($s=p^n$) for s as a complex number. The multiplication does not hold for complex number which insure that division is also not possible. Hence Galois introduced the concept of irreducible polynomial. Further this irreducible polynomial is called minimum function and under this minimum function it follows all the operations of Galois Field.

To do so we consider a function . $a_1\alpha^n + a_2\alpha^{n-1} +\ldots+a_n$.

For different value of a_i and fixed value of α and n we get many functions. The function which can not be factorized is called irreducible polynomial and call such polynomial, a minimum function. Using this function, one can reduce the power of those elements which are greater and equal to n, that is, reduce it up to α^{n-1} . Next take the element up to p^n -2 and find primitive element α such that $\alpha^{p^n} -1 = 1$.

Take all this element and keep them in a key row. Similarly keep these elements in one column also and call this column as a key column

Develop p^n rows and columns using α as a primitive element and minimum function with reduced mod p

Then it will give one MOLS Design To get remaining (s-1) MOLS Design, keep key row as such and change the position of element one by one in a key column and developed it accordingly .

Example : s = 4, $\therefore$ $4=2^2 = p^n$ $\therefore p=2, n=2$.

p = 2 is a prime number and n=2 is a positive integer .

The elements of GF(4) are 0, α^0, α^1, α^2 ..

Let $a_1, a_2, a_3 \in \{0,1\}$ as p = 2. So the function can be written as

$$a_1\alpha^2 + a_2\alpha^1 + a_3\alpha^0 = 1.\alpha^2+1\alpha+1 \{\text{take } a_1= a_2= a_3=1\}$$

$$= \alpha^2+\alpha+1$$

Since this function can not be factorized and hence it is called irreducible polynomial .

Here $\alpha^2+\alpha+1$ is a irreducible polynomial for GF($s=2^2$).

So our minimum function is $\alpha^2+\alpha+1$.

α^0	α^1	α^2
1	α	$\alpha+1$

Now we have to check that is $\alpha^3 = 1$?

$\alpha^3 = \alpha\ \alpha^2 = \alpha(\alpha+1) = \alpha^2 + \alpha = \alpha+1+\alpha = 2\alpha+1 = 1$

since $\alpha^{p^n-1} = 1$ with reduced mod 2 so α is a primitive element of GF(2^2) with the corresponding minimum function $\alpha^2+\alpha+1$.

The elements of GF(4) are 0 1 α α^2

Addition: $\alpha^2+\alpha=1$, subtraction: $\alpha^2-\alpha=1$, multiplication $\alpha^2.\alpha = \alpha^3 = 1$, Division: $\dfrac{1}{\alpha^2} = \dfrac{\alpha^3}{\alpha^2} = \alpha$. Hence it follows the four operations.

First MOLS design is given by

	0	1	α	$\alpha+1$
0	0	1	α	$\alpha+1$
1	1	0	$\alpha+1$	α
α	α	$\alpha+1$	0	1
$\alpha+1$	$\alpha+1$	α	1	0

0	1	α	$\alpha+1$	0	1	α	$\alpha+1$
α	$\alpha+1$	0	1	$\alpha+1$	α	1	0
$\alpha+1$	α	1	0	1	0	$\alpha+1$	α
1	0	$\alpha+1$	α	α	$\alpha+1$	0	1

Example: Consider $s=8 = 2^3 = p^n \Rightarrow p=2, n=3$.
Here p is a prime number and n is an integer. The elements of GF(8) are
$0, \alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$. Since n = 2 so the function can be written as
$a_1\alpha^3 + a_2\alpha^2 + a_3\alpha + a4$
where, $a_1, a_2, a_3, a_4 \in \{0,1\}$
Considering $a_1=0$, $a_2 =0$, $a_3 =1$, $a_4 =1$, we can write given function as
$1.\alpha^3 + 0.\alpha^2 + 1.\alpha + 1 = \alpha^3 + \alpha + 1$. which can not be

factorized and hence it is a irreducible polynomial,

so minimum function is $\alpha^3 + \alpha + 1$.

Now

α^1	α^2	α^3	α^4	α^5	α^6	$\alpha^{8-1} = \alpha^7$
α	α^2	$\alpha + 1$	$\alpha(\alpha^3)$	$\alpha(\alpha^4)$	$\alpha(\alpha^5)$	$\alpha\alpha^6$
			$\alpha(\alpha + 1)$	$\alpha(\alpha^2 + \alpha)$	$\alpha(\alpha^2 + \alpha + 1)$	$= \alpha(\alpha^2 + 1)$
			$\alpha^2 + \alpha$	$\alpha^3 + \alpha^2$	$\alpha^3 + \alpha^2 + \alpha$	$\alpha^3 + \alpha$
				$\alpha^2 + \alpha + 1$	$\alpha + 1 \; + \alpha^2 + \alpha$	$\alpha + 1 + \alpha$
					$\alpha^2 + 1$	1

Here $\alpha^{8-1} = \alpha^{7} = 1$ so α is primitive element with mod 7 with minimum function $\alpha^3 + \alpha + 1$. Now the first MOLS design is

Key	0	1	α	α^2	$\alpha+1$	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	α^2+1
0	0	1	α	α^2	$\alpha+1$	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	α^2+1
1	1	0	$\alpha+1$	α^2+1	α	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	α^2
α	α	$\alpha+1$	0	$\alpha^2+\alpha$	1	α^2	α^2+1	$\alpha^2+\alpha+1$
α^2	α^2	α^2+1	$\alpha^2+\alpha$	0	$\alpha^2+\alpha+1$	α	$\alpha+1$	1
$\alpha+1$	$\alpha+1$	α	1	$\alpha^2+\alpha+1$	0	α^2+1	α^2	$\alpha^2+\alpha$
$\alpha^2+\alpha$	$\alpha^2+\alpha$	$\alpha^2+\alpha+1$	α^2	α	α^2+1	0	1	$\alpha+1$
$\alpha^2+\alpha+1$	$\alpha^2+\alpha+1$	$\alpha^2+\alpha$	α^2+1	$\alpha+1$	α^2	1	0	α
α^2+1	α^2+1	α^2	$\alpha^2+\alpha+1$	1	$\alpha^2+\alpha$	$\alpha+1$	α	0

Example: s=9 $\Rightarrow$ 3^2 $\Rightarrow$ p=3, n=2. The elements of GF(9) are

$0, \alpha^0, \alpha^1, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6, \alpha^7$

Now the function can be written as

$\Rightarrow a_1\alpha^2 + a_2\alpha + a_3$

where, $a_1, a_2, a_3, \in \{0,1,2\}$

Considering $a_1=a_2=1$, $a_3=2$, the function can be written as

$\therefore \alpha^2+\alpha+2$ which can not be factorized so this is a irreducible polynomial

$\therefore$ Hence the minimum function is $\alpha^2+\alpha+2$. for GF(9).

α^0	α^1	α^2	α^3	α^4
0	α	$\alpha^2+\alpha+2$	$\alpha^2(\alpha)$	$\alpha(\alpha^3)$
		$2\alpha^2+\alpha+2$	$(2\alpha+1)\alpha$	$2\alpha^2+2\alpha$
		$\alpha^2+\alpha+2$	$2\alpha^2+\alpha$	$\alpha^2+\alpha+2$
		$3\alpha^2+2\alpha+4$	$\alpha^2+\alpha+2$	$3\alpha^2+3\alpha+2$
		$2\alpha+1$	$3\alpha^2+2\alpha+2$	
			$2\alpha+2$	2

α^5	α^6	α^7	α^8
$\alpha(2)$	$\alpha(2\alpha)$	$\alpha(\alpha+2)$	$\alpha(\alpha+1)$
2α	$2\alpha^2$	$\alpha^2+2\alpha$	$\alpha^2+\alpha$
	$\alpha^2+\alpha+2$	$\alpha^2+\alpha+2$	$\alpha^2+\alpha+2$
	$\alpha+2$	$2\alpha^2+2$	$2\alpha^2+2\alpha+2$
		$\alpha^2+\alpha+2$	$\alpha^2+\alpha+2$
		$\alpha+4$	$3\alpha^2+3\alpha+2$
		$\alpha+1$	1

Here , $\alpha^{9-1} = \alpha^8 = 1$

So α is a primitive element of GF(9).

Key	0	1	α	$2\alpha+1$	$2\alpha+2$	2	2α	$\alpha+2$	$\alpha+1$
0	0	1	α	$2\alpha+1$	$2\alpha+2$	2	2α	$\alpha+2$	$\alpha+1$
1	1	2	$\alpha+1$	$2\alpha+2$	2α	0	$2\alpha+1$	α	$\alpha+2$
α	α	$\alpha+1$	2α	1	2	$\alpha+2$	0	$2\alpha+2$	$2\alpha+1$
$2\alpha+1$	$2\alpha+1$	$2\alpha+2$	1	$\alpha+2$	α	2α	$\alpha+1$	0	2
$2\alpha+2$	$2\alpha+2$	2α	2	α	$\alpha+1$	$2\alpha+1$	$\alpha+2$	1	0
2	2	0	$\alpha+2$	2α	$2\alpha+1$	1	$2\alpha+2$	$\alpha+1$	α
2α	2α	$2\alpha+1$	0	$\alpha+1$	$\alpha+2$	$2\alpha+2$	α	2	1
$\alpha+2$	$\alpha+2$	α	$2\alpha+2$	0	1	$\alpha+1$	2	$2\alpha+1$	2α
$\alpha+1$	$\alpha+1$	$\alpha+2$	$2\alpha+1$	2	0	α	1	2α	$2\alpha+2$

$GF(2^2) = p^n \therefore$ p=2, n=2 where 2 is a prime number.

The minimum function for GF(4) is $\alpha^2 + \alpha + 1$ and the elements are 0, α^0 α^1 α^2

Where, α is primitive element of $GF(2^2)$ so finally the elements of GF(4) are 0,1, α, a+1

$\therefore$ treatment combination .

0	0	1	0	2	0
0	1	1	1	2	1
0	α	1	α	2	α
0	$(\alpha+1)$	1	$(\alpha+1)$	2	$(\alpha+1)$

Example :GF(6) = GF(2×3)

Here 2 and 3 are prime number. The elements of GF(3) and GF(2) are 0,1 and 0,1,2.

Combination		
	0 0	1 0
	0 1	1 1
	0 2	1 0

	0 0	0 1	0 2	1 0	1 1	1 2
0 0	0 0	0 1	0 2	1 0	1 1	1 2
0 1	0 1	0 2	0 0	1 1	1 2	1 0
0 2	0 2	0 0	0 1	1 2	1 0	1 1
1 0	1 0	1 1	1 2	2 0	2 1	2 2
1 1	1 1	1 2	1 0	2 1	2 2	2 0
1 2	1 2	1 0	1 1	2 2	2 0	2 1

This gives first MOLS . Similarly we can obtain other MOLS.

Theorem: For a BIB design prove that 0 <E <1

We know that $E = \dfrac{\lambda v}{rk} = \dfrac{\lambda}{r} \cdot \dfrac{v}{k}$

$$= \frac{k-1}{v-1} \cdot \frac{v}{k} \qquad \begin{cases} \lambda(v-1) = r(k-1) \\ \dfrac{\lambda}{r} = \dfrac{(k-1)}{(v-1)} \end{cases}$$

$$= \frac{k-1}{k} \cdot \frac{v}{v-1} \qquad = \frac{k-1}{k} \Big/ \frac{v-1}{v} = 1 - \frac{1}{k} \Big/ 1 - \frac{1}{v}.$$

since $k > 1 \Rightarrow 0 < 1/k < 1$

$$v > 1 \Rightarrow 0 < 1/v < 1$$

$$1 - 1/k < 1, \; 1 - 1/v < 1$$

In a BIBD k<v $\therefore 1/k > 1/v$

$$\therefore 1 - 1/k < 1 - 1/v$$

$$\therefore \frac{1-{}^{1}\!/_{k}}{1-{}^{1}\!/_{v}} < 1 \quad \therefore E < 1$$

$$\frac{v(\hat{\tau}_i - \hat{\tau}_j)\ RBD}{v(\hat{\tau}_i - \hat{\tau}_j)\ BIBD} < 1.$$

Eigen Roots of C-Matrix of a BIBD

v = 4, b = 6, r = 3, k =2, λ = 1.

$$C = r\, I_v - \frac{NN'}{k}$$

$$= \left. \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix} \middle/ 2 \right.$$

$$= \begin{bmatrix} 3/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 3/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 3/2 & -1/2 \\ -1/2 & -1/2 & -1/2 & 3/2 \end{bmatrix} \qquad (1)$$

Now $C = \theta\,(I_v - 1/v\ E_{vv})$

$$= \theta \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \right]$$

$$= \theta \begin{bmatrix} 3/4 & -1/4 & -1/4 & -1/4 \\ -1/4 & 3/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 3/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 3/4 \end{bmatrix} \qquad (2)$$

Compare (1) and (2), we get $\theta = 2$

Hence the eigen value of C matrix of a BIB design is 2 with multiplicities 3.

SIMPLE LATTICE DESIGN

Fisher and Yate`s (1936) and Bose (1936) introduced and developed a series of IBD which called BIBD with parameters v, b, r, k, λ. Since this design holds when

(1) $b \geq v$, (2) $\lambda (v-1) = r (k-1)$, (3) $vr = bk$.

and hence BIBD are not available with all parameters

Therefore it requires to have another type of IBD.

This incomplete Block Design is called lattice design introduced by Yate`s (1936)

An incomplete block design with parameter v, b, r, k is said to be m – ple lattice design if $v = s^2$ treatments are arranged in m-groups where m-groups form b blocks such that each group contains all the treatment once and only once and each block contain k treat ($k<v$) .However each treatment is repeated r times , If m= 2 then this m – ple lattice design is called either Lattice design or simple lattice design , or square lattice design

If m = 3 then it is called triple lattice design .
If m = r then m-ple, then this Lattice design is called the balanced lattice design . In this case balanced lattice design is a particular case of balanced incomplete block design with parameters $v = s^2$,$b = s (s+1)$, $k = s$,$\lambda = 1$.

METHOD OF CONSTRUCTION

Let there be s^2 treatments numbered by 1.2.3…..s^2 . Let these treatments are arranged in the form of a $s \times s$ square The contents of each of s rows of this square are taken to form a block . Thus s blocks are obtained from s rows. Similarly take treatments from each column and keep them in a block. In this way s more blocks can be obtained from s columns. Again s more block can be obtained from s latin letters. Further more blocks can obtained from another latin square design of same size.

In this way (m-2)s blocks can be obtained from (m-2) orthogonal latin square design. This give a lattice square design with s^2 treatments in ms blocks each of size s and each is replicated m times. Which is called m-ple Lattice square Design. A square Lattice Design with two replications is called a simple Lattice and one with three replication is a triple Lttice.

When s is a prime or a power of a prime, then by using all the (k-1) MOLS for obtaining the blocks as indicated as above a Lattice design in (s+1) replications is obtained. This is called a Balanced Lattice. Balanced Lattice is a particular case of balance incomplete block designs belonging to the series $v = s^2$, $b=s^2+s$, $r=s+1$, $k = s$, $\lambda = 1$.

If the s^2 treatments are coded by the combinations of the s^2 factorial, i.e. ,the combinations of two factors each of k levels then a confounded design in block of size k is obtained by confounding main effects and interactions in m different replication given on m-ple lattice design. These design are therefore also called quasi factorial design. Extending this analogous to factorial design with three factors each at s level, two types of designs corresponding to block size s and s^2 can be obtained by adapting suitable confounding. These design are called cubic Lattice.

ANALYSIS OF LATTICE DESIGN

The data obtained from an m-ple square lattice design is non-orthogonal and are therefore analyzed by method of analysis of non-orthogonal two way data. The linear model of this design is given by :

$$Y_{ijk} = \mu + t_i + b_i + l_{ijk}$$

The reduced normal equation for estimating treatment effect t_i is obtained from

$$c_{ii}t_i + \sum_{m \neq 1} c_{im}t_m = \theta_i$$

where , $c_{ii} = n_i - \sum_j \frac{n_{ij}^2}{n.j}$, $\mathrm{c}_{\mathrm{im}} = -\sum_{\mathrm{j}} \frac{\mathrm{n}_{\mathrm{ij}}.\mathrm{n}_{\mathrm{mj}}}{\mathrm{n.j}}$

The incidence matrix of the design is shown bellow.

	Treatment					blocks size(n_j)	Block total
	1	2	3	$\cdots$	k^2		
1	1	1	0	$\cdots$	1	k	B_1
2	0	0	1	$\cdots$	1	k	B_2
3	1	1	0	$\cdots$	0	k	B_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$		$\vdots$	$\vdots$
mk	0	1	0	$\cdots$	1	k	B_{mk}
Rep.	m	m	m	…	m		
Tr.T	T_1	T_2	T_3	…	T_{k^2}		
Ad.T	Q_1	Q_2	Q_3	…	Q_{k^2}		

normal equation of t_i is given as:

$$(m-\frac{m}{k})t_1-\frac{1}{k}(t_2+t_3+t_4+t_6+\ldots)-\frac{0}{k}(t_5+t_9+\ldots)=Q_1$$

$$mt_1 - \frac{1}{k}(t_1+t_2+t_3+\ldots\ldots.)-\frac{1}{k}(t_1+t_4+t_7+\ldots\ldots.)$$

$$-\frac{1}{k}(t_1+t_6+t_8+\ldots\ldots.)-\frac{0}{k}(t_1+t_5+t_9+\ldots\ldots.)=Q_1$$

$$\therefore mkt_1 - (t_1+t_2+t_3+\ldots\ldots.)-(t_1+t_4+t_7+\ldots\ldots.)-$$

$$(t_1+t_6+t_8+\ldots\ldots.)=Q_1k$$

$$\therefore mkt_1 - S_R(t_1) - S_C(t_1) - S_P(t_1) = kQ_1$$

Where, $S_R(t_1)$ is sum of all those treatments which occur in row block having treatment 1. $S_C(t_1)$ is sum of all those treatments which occur in column block having treatment 1. $S_P(t_1)$ is sum of all those treatments which occur in (m-2) MOLS design.

Now, the normal equation for t_i is given by

$$\therefore mkt_i - S_R(t_i) - S_C(t_i) - S_P(t_i) = k\,Q_i\,,\ \ i = 1,2,\ldots.k^2 \quad (1)$$

Writing all equation like (1) for each of the treatment t_i present in $S_R(t_i)$ and adding these k equation we get $mkS_R(t_i) - kS_R(t_i) - \sum_{i=1}^{k^2} t_i - \sum_{i=1}^{k^2} t_i \ldots = k\,S_R(Q_i)$

where $S_R(Q_i)$ is the adjusted total of those treatments which are present in $S_R(t_i)$. Taking $\sum_i t_i = 0$ we get

$$\therefore mkS_R(t_i) - kS_R(t_i) = kS_R(Q_i)$$

$$\therefore (mk-k)S_R(t_i) \quad = kS_R(Q_i)$$

$$\therefore (m-1)S_R(t_i) \quad = S_R(Q_i)$$

$$\left.\begin{aligned} \mathrm{S_R}(\mathrm{t_i}) &= \frac{\mathrm{S_R}(\mathrm{Q_i})}{\mathrm{m}-1} \\ \mathrm{S_C}(\mathrm{t_i}) &= \frac{\mathrm{S_C}(\mathrm{Q_i})}{\mathrm{m}-1} \\ \mathrm{S_P}(\mathrm{t_i}) &= \frac{\mathrm{S_P}(\mathrm{Q_i})}{\mathrm{m}-1} \end{aligned}\right\} \qquad (2)$$

On putting the value of (2) in (1) we get ,

$$mkt_i - \frac{S_R(Q_i)}{m-1} - \frac{S_C(Q_i)}{m-1} - \frac{S_P(Q_i)}{m-1} = kQ_i$$

Hence the solution of the estimate of treatment effect t_i is obtained as

$$\hat{t}_i = \frac{kQ_i}{mk} + \frac{S_R(Q_i)}{mk(m-1)} + \frac{S_C(Q_i)}{mk(m-1)} + \frac{S_P(Q_i)}{mk(m-1)} \quad (3)$$

Similarly, the solution of the estimate of treatment effect t_i' is given by

$$\hat{t}_i' = \frac{kQ_i'}{mk} + \frac{S_R(Q_i')}{mk(m-1)} + \frac{S_C(Q_i')}{mk(m-1)} + \frac{S_P(Q_i')}{mk(m-1)} \quad (4)$$

Now the difference between effects of the i[th] and i' th treatment is given by.

$$\hat{t}_i - \hat{t}_i' = \frac{Q_i - Q_i'}{m} + \frac{S_R(Q_i) - S_R(Q_i')}{mk(m-1)} +$$

$$\frac{S_C(Q_i) - S_C(Q_i')}{mk(m-1)} + \frac{S_P(Q_i) - S_P(Q_i')}{mk(m-1)}$$

Variances of the treatment contrasts:

Case–1. If the i[th] and j[th] treatments occur together in the same block then $S_R(Q_i)$ and $S_R(Q_i')$ are identical and hence $S_R(Q_i)$ -$S_R(Q_i')$ = 0

$$\therefore V(\hat{t}_i - \hat{t}_i') = (\frac{1}{m} + \frac{1}{m})\sigma^2 + \frac{2(m-1)}{mk(m-1)}\sigma^2$$

$$= \frac{2}{m}\sigma^2 + \frac{2}{mk}\sigma^2 = \frac{2}{m}(1+\frac{1}{k})\sigma^2 \qquad \text{(A)}$$

Case 2: Let t_i and t_i' are those treatments which do not occur in the some block in which t_i and t_i' occur.

$$\therefore V(\hat{t}_i - \hat{t}_i') = \frac{2}{m}\sigma^2 + \frac{2m\sigma^2}{mk(m-1)}$$

$$= (2/\mathrm{m})\left(1 + \frac{\mathrm{m}}{\mathrm{k}(\mathrm{m}-1)}\right)\sigma^2 \qquad \text{(B)}$$

Average Variance of M-ple Lattice Design

Let v_1 is the variance of those treatment contrasts which occur together in the block of same row or same column .

Let v_2 is the variance of those treatment contrasts which do not occur together in the block of same row or same column.

Let $n_1 = m (k-1)$ = the number of treatments each of which occur with, say, t_i in same block or in other blocks.

Let $n_2 = (k-1)(k+1-m)$ =number of treatments which do not with t_i in any block . Hence average variance can be obtained as :

Relative Efficiency of m-ple Lattice Design.

$$E = \frac{v(t_i - t_i')\ RBD}{v(t_i - t_i')\ Lattice\,design}$$

$$= \frac{2}{m}\sigma^2 \Bigg/ \frac{n_1 v_1 + n_2 v_2}{n_1 + n_2}$$

$$= \frac{2(n_1 + n_2)}{m(n_1 v_1 + n_2 v_2)}\sigma^2$$

YOUDEN SQUARE DESIGN

BIBD is available only for require number of parameters because of their parametric relations. Similarly Lattice design is available only for perfect square number of treatments or cubic number of treatments . However it requires large number of unit (plot) for small number of size which will cause of huge cost and high expenditure. This shows that Balanced incomplete block design and Lattice design are not available for specified number of v, b, r, k. Therefor it required to have another type of Incomplete block design which is called Youden Square Design. This design is introduced by Youden and hence the nomenclature.

DEFINITION:

An incomplete block design is said to be Y.S.D. if v treatments are arranged in b block where each block is repeated r times provided block of YSD from BIBD such that

- Every row contains all the treatments once, i.e., row wise it is a complete block design.

(ii) Column wise it is a symmetrical BIBD.

METHOD OF CONSTRUCTION

First of all construct a LSD or an orthogonal Latin Square Design of size s . Delete more than one row such that the column forms BIBD. For the given LSD, a YSD can always be obtained provided the columns of the LSD forms of BIBD.

EXAMPLE : LSD of size 4

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ \hline 4 & 1 & 2 & 3 \\ \hline \end{array} \qquad \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{array} \qquad \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \\ 4 & 1 & 2 \end{array}$$

This is YSD with parameters v=b= , r=k=3, λ=2.

Example: LSD of size 3

1	2	3		2	3
2	3	1		3	1
3	1	2		1	2

This is YSD with parameters v=b=3, r=k=2, λ=1.

Exam: LSD of size 7

1	2	3	4	5	6	7		1	2	4
2	3	4	5	6	7	1		2	3	5
3	4	5	6	7	1	2		3	4	6
4	5	6	7	1	2	3		4	5	7
5	6	7	1	2	3	4		5	6	1
6	7	1	2	3	4	5		6	7	2
7	1	2	3	4	5	6		7	1	3

Deleting 3^{rd}, 5^{th}, 6^{th}, and 7^{th} , we get a Youden square design. If we write the design column wise, we get a BIB design with parameters v=b=7, r=k= 3 and λ= 1.

Comparison between Youden Square Design and Latin square Design.

Y.S.D..	L.S.D.
It is an incomplete Block Design	It is a complete Block Design
It is an incomplete L.S.D	it is complete L.S.D.
Column of YSD from BIBD	Column of LSD from RBD
All the YSD are incomplete Latin square design.	All the LSD does not give YSD.

Analysis of Youden Square Design

YSD is a design which eliminates heterogeneity on both direction and hence analysis of YSD can be carried out similar to the analysis of two way elimination of heterogeneity . The reduced normal equation of two way analysis of elimination of heterogeneity is given by

$$Q = T - \frac{LR}{u'} - \frac{MC}{u} + \frac{G\,L\,E_{u1}}{uu'} \qquad (1)$$

$$F = \mathrm{diag}(r_1,\ldots,r_v) - \frac{LL'}{u'} - \frac{MM'}{u} + \frac{L\,E_{uu}\,L'}{uu'} \qquad (2)$$

Since the row contain all the treatments and hence the incidence matrix $L = l_{ij} = 1$ for $i \neq j = 1,2\ldots v = E_{vr}$.

Similarly the column forms the BIB design so the incidence matrix $M = m_{ik} = 1$, if i^{th} treatment occur in k^{th} block, otherwise $=0$.

$M = m_{ik} = n_{ij} = N$ as the column forms a BIB design due to column from BIBD, $u = k = r$, $v = b = u'$, $C=B$.

Now from (1) we can rewrite as

$$Q = \underline{T} - \frac{E_{vr}R}{v} - \frac{NC}{k} + \frac{GE_{vr}E_{r1}}{rv}$$

$$Q = \underline{T} - \frac{GE_{v1}}{v} - \frac{NB}{k} + \frac{GrE_{v1}}{vr}$$

$$= \underline{T}\ \frac{GE_{v1}}{v} - \frac{NB}{k} + \frac{GE_{v1}}{v} \qquad = \underline{T} - \frac{\mathrm{NB}}{\mathrm{k}}$$

where C = B as column of youden square design gives the block of SBIB design, i.e., columns of Youden square design to be considered as blocks of SBIB design. This is the same as the adjusted treatment total in case of BIB design.

Now,

$$F = \text{diag}(r_1 \ldots . r_v) - \frac{LL'}{u'} - \frac{MM'}{u} + \frac{LE_{uu}L'}{uu'}$$

Since, $r_1 = r_2 = r_v = r$, N=M, u=r, $L=E_{vr}$, $L'=E_{rv}$, N'=M', u'=v.

$$F = \text{diag}(r \ldots . r) - \frac{\text{Evr Erv}}{v} - \frac{NN'}{k} + \frac{E_{vr}E_{rr}E_{rv}}{rv}$$

$$= rI_v - \frac{\text{rEvv}}{v} - \frac{NN'}{k} + \frac{r\text{EvrErv}}{rv}$$

$$= rI_v - \frac{\text{rEvv}}{v} - \frac{NN'}{k} + \frac{r\text{Evv}}{v}$$

$$= rI_v - NK^{-1}N' \qquad (3)$$

which is the C-matrix of a SBIB Design.

From (3) $\text{F} = rI_v - \dfrac{NN'}{k}$

$NN' = (r-\lambda)I_v + \lambda E_{vv}$. Now putting this in (3), we get

$$\text{F} = rI_v - \left[\frac{(r-\lambda)I_v + \lambda E_{vv}}{k}\right] \quad = \left[r - \frac{(r-\lambda)}{k}\right]I_v - \frac{\lambda}{k}E_{vv}$$

$$= \left[\frac{rk - r + \lambda}{k}\right]I_v - \frac{\lambda}{k}E_{vv} \quad = \left[\frac{r(k-1)+\lambda}{k}\right]I_v - \frac{\lambda}{k}E_{vv}$$

$$= \left[\frac{\lambda(v-1)+\lambda}{k}\right]I_v - \frac{\lambda}{k}E_{vv} \quad = \frac{\lambda v}{k}I_v - \frac{\lambda}{k}E_{vv}$$

$$\therefore F = \frac{\lambda v}{k}\left[I_v - \frac{E_{vv}}{v}\right]$$

Now, $Q = F\hat{\tau} = \dfrac{\lambda v}{k}\left[I_v - \dfrac{E_{vv}}{v}\right]\hat{\tau} = \left[\dfrac{\lambda v I_v}{k} - \dfrac{\lambda E_{vv}}{k}\right]\hat{\tau}$

$$= \frac{\lambda v \hat{\tau}}{k} - \frac{\lambda E_{vv}\hat{\tau}}{k}$$

$$Q = \frac{\lambda v \hat{\tau}}{k} \quad \left\{ \begin{aligned} E_{vv}\hat{\tau} &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & \cdots & 1 \end{pmatrix}_{v\times v} \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_v \end{pmatrix}_{v\times 1} \\ &= \begin{pmatrix} \tau_1 + \cdots + \tau_v \\ \vdots \quad \vdots \quad \vdots \\ \tau_1 + \cdots + \tau_v \end{pmatrix}_{v\times v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{aligned} \right.$$

$$\hat{\tau} = \frac{k}{\lambda v} Q$$

Now , Treatment sum of squares $= \hat{\tau}' Q = \dfrac{k}{\lambda v} Q' Q$

ANOVA TABLE

S.V.	d.f.	Sum of square	mean squares	F-ratio
Treatment (adjusted)	v-1	$\frac{k}{\lambda v}\sum_{i=1}^{v} Q_i^2 = S_t^2$	$S_t^2 / v-1$ (=M_1)	M_1 / σ^2
Block (unadjusted)	r-1	$\sum_{j=1}^{b} B_j^2 / n.j - CF$	$S_{bj}^2 / r-1$ (=M_2)	M_2 / σ^2
Error	vr – v – r + 1	s^2	s^2 / (vr-v-r+1) = σ^2	
Total	vr – 1	y'y		