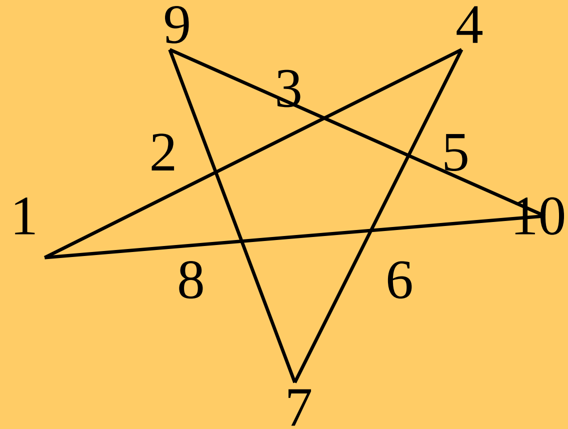
PARTIALLY BALANCED INCOMPLETE BLOCK DESIGN

BIBD is not available for all treatments as it has to satisfy the following condition $vr = bk$, $\lambda(v-1) = r(k-1)$, $b \geq v$.

Again lattice design is available only for the square number of treatments and cubic number of treatments while youden square design required large number of replication size. So to have an IBD with all possible treatments and a smaller replication size, we required another class of incomplete block design. For such design Bose and Nair (1936) introduce the concept of PBIBD. Which is available for all possible number of treatments and a smaller number of replication size.

Association Schemes



1^{st} *associate*	2^{nd} *associate*
$1 \rightarrow 2,3,4,6,8,10$	5,7,9
$2 \rightarrow 1,3,4,7,8,9$	5,6,10
$3 \rightarrow 1,2,4,5,9,10$	6,7,8
$4 \rightarrow 1,2,3,5,6,7,$	8,9,10
$5 \rightarrow 3,4,6,7,9,10$	1,2,8
$6 \rightarrow 1,4,5,7,8,10$	2,3,9
$7 \rightarrow 2,4,5,6,8,9$	1,3,10
$8 \rightarrow 1,2,6,7,9,10$	3,4,5
$9 \rightarrow 2,3,5,7,8,10$	3,4,5
$10 \rightarrow 1,3,5,6,8,9$	2,4,7
$\therefore n_1 = 6$	$n_2 = 3$

$\therefore\ n_1 + n_2 = 6+3 = 9$

$v-1 = 10-1 = 9$

$\therefore\ n_1 + n_2 = v-1$.

Let there are v treatments denoted as 1,2, ...,v. These treatments follow association scheme, if it satisfies the following:

(i)For a given treatment θ, there are i^{th} associate treatments.

(ii) For a given treat θ, i^{th} associate treatments occur n_i times

(iii) Pair of treatments occurs together λ_i times. For any two treatment, say, θ and ϕ, number of common treatment between i^{th} associate of θ and j^{th} associate of ϕ is constant and is denoted by p_{ij} matrix which is given by

$$\therefore P_{ij}^{k} = \begin{pmatrix} P_{11}^{k} & \cdots & P_{1m}^{k} \\ \cdots & \cdots & \cdots \\ P^{k}m1 & \cdots & P_{mm}^{k} \end{pmatrix}$$

where P^1_{ij} and $P_{ij}^{\ 2}$ … , (k= 1,2,…, m) are called association matrix of 1st and 2nd …, m th associate classes and the whole scheme is called association scheme.

Suppose treatments 1 and 2 are 1st associate then .

$\therefore p'_{11} = 3,\ p'_{12} = 2,\ p'_{21} = 2,\ \ p^{2}_{22} = 1.$

$1 \rightarrow 2,3,4,6,8,10$ $\qquad$ $2 \rightarrow 1,3,4,7,8,9$

$1 \searrow 5,7,9$ $\qquad$ $2 \searrow 5,6,10.$

$\therefore p'_{11} =$ number of common treatment between 1(1), and 2(1) = {3,4,8}=3

$\therefore p'_{22} =$ number of common treatment between 1(2), and 2(2) = {5}=1

$p^{(1)}_{12} = p^{(1)}_{21} =$ number of common treatment between 1(2) and 2(1) = {7,9} = 2.

1(1) and 2(2) common ={6,10} = 2

$$\therefore\ p'_{ij} = \begin{pmatrix} p'_{11} & p'_{12} \\ p'_{21} & p'_{22} \end{pmatrix} \therefore\ p'_{12} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

Now consider treatments 1 and 5 are 2^{nd} associate .

(1) $\quad 1 \rightarrow (2,3,4,6,8,10)$ A $(5,7,9)$ C

$\quad 5 \rightarrow (4,6,7,3,9,10)$ B $(1,2,8)$ D

$\therefore$ $\quad 2^{nd}$ associate

$$p^2_{11} = A \cap B = \{6,10,3,10\} = 4$$

$$p^2_{12} = A \cap D = \{2,8\} = 2$$

$$p^2_{21} = B \cap C = \{7,9\} = 2 \qquad \therefore\ p^2_{12} = p^2_{21}$$

$$p^2_{22} = C \cap D = \{\phi\} = 0$$

$$\therefore\ p^2_{ij} = \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}$$

This shows that associate matrix are symmetric matrix

Now, $p_{11}^{1}+p_{12}^{1}+p_{21}^{1}+p_{22}^{1}=3+2+2+1=8$

$$\therefore \sum_{i=j=1}^{2} p_{ij}^{1}=v-2$$

also $$\therefore \sum_{i=j=1}^{2} p_{ij}^{2}=v-2$$

$$\therefore \sum_{i=1}^{m} n_i=v-1$$

Definition:

Partially Balanced Incomplete Block Design (PBIBD)

An incomplete block design is said to be PBIB Design if v treatments are arranged in b blocks, each block contains k treatments ($k<v$), each treatment occur in r blocks and a pair of treatments occur together in λ_i blocks ($i = 1,2,\ldots,m$), provided it follows the following association schemes .

- In association scheme, there are i^{th} classes .
- For a given treatment, say, θ, n_i treatments occur in i^{th} associate class
- For any given treatment, say, θ, the remaining treatments, if they are i^{th} associate class, occur together in λ_i blocks.
- For any two treatments, say, θ and ϕ, number of common treatment between i^{th} associate of θ and j^{th} associate of ϕ is constant and is denoted by p_{ij} matrix.

Relation between BIBD and PBIBD.

If $\lambda_1=\lambda_2=...=\lambda_i=\lambda$ then PBIBD becomes BIBD.

Difference between BIBD and PBIBD.

BIBD	PBIBD
Pair of treatments occur λ times	Pair of treatments occur λ_i times
$b \geq v$	dose not hold.
dose not hold association schemes	Hold association schemes.

Parameters of PBIBD.

On the basis of association schemes, PBIBD has two types of parameters.

(1) Primary parameters: v, b, r, k, λ_i $(i = 1,2,\ldots,m)$

(2) Secondary parameters: n_i, p_{ij}^k $i \neq j = (1,2,\ldots,v)$

$k = 1,2,\ldots,m$

Parametric Relation.

(1) $vr = bk$ (ii) $\sum_{i=1}^{n} n_i = v-1$ (iii) $\sum_{i=1}^{m} n_i\lambda_i = r(k-1)$

prove that $\sum n_i = v-1$

For any PBIB Design with m[th] association schemes

we know that $\sum_{i=0}^{m} B_i = E_{vv}$

now post multiply both said by E_{v1}

$$\sum_{i=0}^{m} B_i E_{v1} = E_{vv} E_{v1}$$

Where B_i is a matrix of order v×v and is called association matrix since every associate, i.e., i^{th} associate has n_i treatment.

$$\therefore \sum_{i=0}^{m} n_i E_{v1} = v E_{v1} \Rightarrow \sum_{i=0}^{m} n_i = v$$

$$\Rightarrow n_o + \sum_{i=1}^{m} n_i = v \qquad where\ \ n_0 = 1$$

$$\Rightarrow \sum_{i=1}^{m} n_i = v-1$$

$$E_{vv}E_{v1} = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}_{v\times v} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{v\times 1} = \begin{pmatrix} v \\ \vdots \\ v \end{pmatrix}_{v\times 1} = v\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = vE_{v1}$$

Prove that: $\sum_{i=1}^{m} n_i\lambda_i = r(k-1)$

For any PBIB Design with m^{th} association scheme,

we know that: $\sum_{i=0}^{m} B_i = E_{vv}$ (1) and

$$NN' = \sum_{i=0} \lambda_i B_i \quad (2)$$

Where λ_i is number of times a pair of treatments occur together in i^{th} associate class. Now post multiply by E_{v1} of both said of (2).

$$\therefore NN'E_{v1} = \sum_{i=0}^{m} \lambda_i B_i E_{v1} \quad \therefore N(N'E_{v1}) = \left(\sum_{i=0}^{m} \lambda_i B_i \right) E_{v1}$$

$$\therefore NkE_{b1} = \sum_{i=0}^{m} \lambda_i n_i E_{v1} \text{ or, } kNE_{b1} = \sum_{i=0}^{m} n_i \lambda_i E_{v1}$$

$$\therefore krE_{v1} = \sum_{i=0}^{m} n_i \lambda_i E_{v1} \Rightarrow kr = \sum_{i=0}^{m} n_i \lambda_i$$

$$\Rightarrow kr = n_0 \lambda_0 + \sum_{i=1}^{m} n_i \lambda_i$$

$$\Rightarrow \sum_{i=1}^{m} n_i \lambda_i = kr - n_0 \lambda_0 \quad = kr - r \quad = r(k-1)$$

$$\Rightarrow \sum_{i=1}^{m} n_i \lambda_i = r(k-1)$$

Example :-

$v = b = 4$, $r = k = 2$, $\lambda = 1$.

	1^{st} *associate*	2^{nd} *associate*
1 2	$1 \rightarrow 2,4$	3
2 3	$2 \rightarrow 1,3$	4
3 4	$3 \rightarrow 2,4$	1
4 1	$4 \rightarrow 1,3$	2

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ B_0 & B_1 & B_2 \end{array}$$

Here , $n_1 = 2$ $\therefore \lambda_1 = 1$, $n_2 = 1$ $\lambda_2 = 0$

In B_0, B_1 & B_2 if the treatment is present, write1, otherwise 0 .

$$B_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 4} \quad B_1 = \begin{bmatrix} 2 & 4 \\ 1 & 3 \\ 2 & 4 \\ 1 & 3 \end{bmatrix} = \begin{array}{|cccc} 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array}$$

second associate

$$B_2 = \begin{pmatrix} 3 \\ 4 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}_{4\times 4} \qquad \therefore B_0 = I_4$$

Now $B_1 + B_2 + B_3 =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\therefore \sum_{i=0}^{2} B_i = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}_{4\times 4} = E_{44} = E_{vv}$$

$$\therefore \sum_{i=0}^{m} B_i = E_{vv}$$

Now we want to prove, $NN' = \lambda_0 B_0 + \sum_{i=1}^{m} \lambda_i B_i$

Now $\lambda_1 B_1 + \lambda_2 B_2 =$

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\therefore \sum_{i=1}^{2} \lambda_i B_i = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\lambda_0 B_0 = r B_0 = 2 B_0 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

now , $\lambda_0 B_0 + \sum_{i=1}^{2} \lambda_i B_i$

$$= \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \qquad (1)$$

Here , Incident matrix N =

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 1 & 0 \\ 4 & 0 & 0 & 1 & 1 \end{array}$$

$$N' = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad \therefore NN' = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix} \qquad (2)$$

from (1) and(2)

$$NN' = \lambda_0 B_0 + \lambda_1 B_1 + \lambda_2 B_2 \qquad here, \lambda_0 = r \quad always$$

$$= rB_0 + \sum_{i=1}^{m} \lambda_i B_i$$

Classification of PBIB design:

Bose and Simamoto (1952) classified PBIB Design of two associate classes in to following types on the basis of its association schemes.

- Simple PBIB Design with $\lambda_1 = 0$ or $\lambda_2 = 0$.
- Group divisible design.
- Rectangular type PBIB Design.
- Latin square type PBIB Design.
- Cycle PBIB Design.

Again Bose introduced another type of PBIB Design with two associate classes and named it as Partial geometry type PBIBD. The remaining PBIB Design which do not fall under these 6 categories on the basis of their association schemes & other parameters are called PBIB Design of miscellaneous type.

Simple PBIBD Design:

A PBIB design with two associate classes is said to be simple PBIB Design, if either (i) $\lambda_1 \neq 0$, $\lambda_2 = 0$ or (ii) $\lambda_1 = 0$, $\lambda_2 \neq 0$.

Group Divisible Design. (GDD):

GD Design is simplest class of PBIB Design. A PBIB Design is called GD Design if v = mm treatments are grouped in to m groups each of n treatments such that the treatment belonging to the same group is called 1st associate treatment and treatment belonging to different group are called 2nd associate treatments.

The following are the parameters of GD Design.

$v, b, r, k, \lambda_1, \lambda_2, \quad p_{jk}, m, n$

Example- $\begin{matrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{matrix} \left\{ \begin{array}{l} \textit{here row wise all pair is called } 1^{st} \textit{ associate but if we} \\ \textit{take } (1,4) \textit{ which is different group so it is } 2^{nd} \textit{ associate.} \end{array} \right.$

Here, m = 3 (3-rows), n = 2 (each row has 2 treatments)

v = 6 = 3×2 = 6.

So v = mn , $\therefore$ this is GD Design

Parametric relation:

$n = n_1 + 1 \qquad\qquad n_1 = n-1$

$m = (n_2/n) + 1 \qquad\qquad n_2 = n(m-1)$

$$p_{jk}^{1} = \begin{pmatrix} n-2 & 0 \\ 0 & n(m-1) \end{pmatrix}, \qquad p_{jk}^{2} = \begin{pmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{pmatrix},$$

Bose and Connor (1952) characterized group divisible deign in to three category on the basis of characteristic root of NN' matrix of GD Design.

(1) Singular group divisible design .(SGD Design).

(2) Semi regular group divisible design .(SRGD Design).

(3) Regular group divisible design .(RGD Design).

Singular group divisible design(SRGD)

A GD design is called singular group divisible if it satisfy the following characteristic roots of NN' matrix.

(i) $r-\lambda_1 = 0$ and $rk - v\lambda_2 > 0$,

otherwise, non singular group divisible designs.

Non singular group divisible designs are either semi regular group divisible or regular group divisible designs.

Semi Regular group divisible design(SRGD):
A GD design is said to be SRGD design if the characteristic root of NN' matrix satisfy the following:

(i) $r-\lambda_1 \neq 0$ (ii) $rk - v\lambda_2 = 0$.

Regular group divisible design(RGD design):
A GDD is said to be RGD Design if the characteristics root of NN' matrix satisfy the following condition .

(i) $r-\lambda_1 > 0$ (ii) $rk - v\lambda_2 > 0$.

Example: Identify the design.Write its parameter, obtain

C-matrix, NN' matrix ,association matrix and show that.

$$NN' = \sum_{i=0}^{2} B_i \lambda_i \qquad \sum_{i=0}^{2} B_i = E_{vv}$$

$$\begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \\ 4 & 5 & 1 \\ 5 & 6 & 2 \\ 6 & 1 & 3 \end{array}$$

$\rightarrow$ *This is an Incomplete Block Design*

$\rightarrow$ $v = b = 6, \quad r = k = 3$

This is a symmetrical IBD

$\rightarrow$ $\lambda_1 = 1, \lambda_2 = 2, n_1 = 4, n_2 = 1$

1 → 2,3,5,6, n_1
1 → 4 n_2

2 → 1,3,4,6
2 → 5

$$\therefore p_{ij}^{1} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \therefore p_{ij}^{2} = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$$

This is a PBIBD of two associate classes

$n = n_1 + 1 \qquad \therefore n = 5,$

$m = \dfrac{n_2}{n} + 1 \qquad m = \dfrac{1}{5} + 1 = \dfrac{6}{5}$

v = nm =5. 6/5 = 6

this is not a group divisible design.

$r - \lambda_1 = 1 = 2 > 0.$

$rk - v\lambda_2 = 9$

Incidence matrix N

$$N = \begin{array}{c|cccccc} T \diagdown B & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 1 & 0 & 0 & 1 \\ 4 & 1 & 0 & 1 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 & 1 & 1 & 0 \\ 6 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}$$

$$N' = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad NK^{-1} = \begin{bmatrix} 3 & 1 & 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 & 1 & 1 \\ 1 & 2 & 1 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 & 1 & 3 \end{bmatrix}_{6\times 6}$$

C-matrix $\quad C = rI_v - NK^{-1}N' = rI_v - \dfrac{NN'}{k}$

$$C = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 1 & 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 & 1 & 1 \\ 1 & 2 & 1 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 & 1 & 3 \end{bmatrix} \Big/ 3$$

$$\therefore C = \begin{bmatrix} 2 & -1/3 & -1/3 & -2/3 & -1/3 & -1/3 \\ -1/3 & 2 & -1/3 & -1/3 & -2/3 & -1/3 \\ -1/3 & -1/3 & 2 & -1/3 & -1/3 & -2/3 \\ -2/3 & -1/3 & -1/3 & 2 & -1/3 & -1/3 \\ -1/3 & -2/3 & -1/3 & -1/3 & 2 & -1/3 \\ -1/3 & -1/3 & -2/3 & -1/3 & -1/3 & 2 \end{bmatrix}$$

$$\lambda_1 B_1 = 1 \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\lambda_2 B_2 = 2\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\lambda_0 B_0 = 3.B_0 = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\text{now},\ \lambda_0 B_0 + \lambda_1 B_1 + \lambda_2 B_2 = \begin{bmatrix} 3 & 1 & 1 & 2 & 1 & 1 \\ 1 & 3 & 1 & 1 & 2 & 1 \\ 1 & 1 & 3 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 & 1 & 1 \\ 1 & 2 & 1 & 1 & 3 & 1 \\ 1 & 1 & 2 & 1 & 1 & 3 \end{bmatrix} = NN'$$

$$\therefore \sum_{i=0}^{2} \lambda_i B_i = NN'$$

$$\text{now},\ B_0 + B_1 + B_2 = \sum_{i=0}^{2} B_i = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = E_{vv}$$

$$\therefore \sum_{i=0}^{2} B_i = \quad E_{vv}$$

further C-matrix can be written as

$$3C = \begin{bmatrix} 6 & -1 & -1 & -2 & -1 & -1 \\ -1 & 6 & -1 & -1 & -2 & -1 \\ -1 & -1 & 6 & -1 & -1 & -2 \\ -2 & -1 & -1 & 6 & -1 & -1 \\ -1 & -2 & -1 & -1 & 6 & -1 \\ -1 & -1 & -2 & -1 & -1 & 6 \end{bmatrix}$$

$$\text{Now} \begin{pmatrix} 6 & -1 & -1 \\ -1 & 6 & -1 \\ -1 & -1 & 6 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= 7I_3 - E_{33}$$

similarly

$$\begin{pmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{pmatrix} = -2I_3 + I_3 - E_{33}$$

$$\therefore \mathrm{C} = \begin{bmatrix} 7\mathrm{I}_3 - \mathrm{E}_{33} & -\mathrm{I}_3 - \mathrm{E}_{33} \\ -\mathrm{I}_3 - \mathrm{E}_{33} & 7\mathrm{I}_3 - \mathrm{E}_{33} \end{bmatrix} \Big/ 3$$

1. Eigenvalue of C-matrix = 7/3 with multiplicity (3-1) = 2
2. Eigenvalue of C-matrix = 7/3 with m =2

$\therefore Q_1 = Q_2 = Q_3 = Q_4 = 7/3.$

Other eigenvalue will be obtained by solving this matrix.

If we consider $n_1 = 1$ and $n_2 = 4$ then n =2 and m = 3 so it becomes group divisible designs. Again $r - \lambda_1 = 3 – 2 > 0$ and $rk – v\,\lambda_2 > 0$, so design is regular group divisible design.

Triangular TYPE PBIB Design.

A PBIB design with two associate classes is said to be Triangular, if the number of treatments v = n(n-1)/2 and the association scheme is arranged in n rows and n columns such that :

(i) The position in the principle diagonal of the scheme are left blank.

(ii) The n(n-1) /2 positions above the principal diagonal are filled by the treatment numbers 1,2,…,n(n-1)/2

(iii) The n(n-1)/2 position bellow the diagonal are so filled that the array is symmetrical about the principal diagonal, and

(iv) For any treatment i the first associates are exactly those treatment which lie in the same row as i.

$$n_1 = 2n-4; \qquad n_2 = (n-2)(n-3)/2$$

$$p_{ij}^{1} = \begin{bmatrix} (n-2) & (n-3) \\ (n-3) & (n-3)(n-4)/2 \end{bmatrix}$$

$$p_{ij}^{2} = \begin{bmatrix} 4 & 2n-8 \\ 2n-8 & (n-4)(n-5)/2 \end{bmatrix}$$

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$n_1 = 2n-4; \qquad n_2 = (n-2)\,(n-3)/2$

$$p_{ij}^{1} = \begin{bmatrix} (n-2) & (n-3) \\ (n-3) & (n-3)(n-4)/2 \end{bmatrix}$$

$$p_{ij}^{2} = \begin{bmatrix} 4 & 2n-8 \\ 2n-8 & (n-4)(n-5)/2 \end{bmatrix}$$

LATIN SQUARE TYPE OF PBIBD :-

Let a square array of n rows and n columns be formed with n^2 treatments, numbering from 1 to n^2 so that two treatments are first associate if they occur in the same row or the in same column of the array and the second associates otherwise.

A design with the above array as association scheme is said to belong to the subtype L_2 .

Subtype L_3: If one can form a square array of n^2 treatments numbers from 1 to n^2 and to impose a latin square with n letters on this array, so that any two treatments are first associate if they occur in the same row or column of the array or correspondence to the same letter of the latin square and are second associates otherwise.

In this design the secondary parameters are

$$\therefore n_1 = L(n-1), \qquad n_2 = (n-1)(n-L+1)$$

$$p_{ij}^{1} = \begin{array}{cc} L^2 - 3L + n & (L-1)(n-L+1) \\ (L-1)(n-L+1) & (n-L)(n-L+1) \end{array}$$

$$p^{2}_{ij} = \begin{array}{cc} L(L-1) & L(n-L) \\ L(n-L) & (n-L)^2 + (L-2) \end{array}$$

where L = 2 and L = 3 for sub type L_2 and L_3 respectively.

Cyclic PBIB Design

A non group divisible PBIB Design is called cyclic PBIBD if the set of first associates of the treatment numbered i is obtained by adding i-1 to the numbers in the set of first associates of the treatment numbered 1 and subtracting v whenever the sum exceeds v.

Example: Find out eigen value of example 1 write its parameters The given design is

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \\ 4 & 1 \end{bmatrix}$$

$n_1 = 1, \lambda_1 = 0, \; n_2 = 2 , \lambda_2 = 1$

$$p_{ij}^{1} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \qquad p_{ij}^{2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$v = b = 4, \quad r = k = 2.$

$n = n_1 + 1 = 1+1 = 2$

$$m = \frac{n_2}{n} + 1 = \frac{2}{2} + 1 = 2$$

$v = mn = 2x2 = 4$ so the design belong to GD design.

$r - \lambda_1 = 2 - 0 = 2 > 0$

$rk - v\lambda_2 = 2(2) - 4(1) = 0$

This design is semi regular group divisible design

We have already proved that,

$$NN' = \sum_{i=0}^{2} B_i \lambda_i$$

Now C-matrix is given by

$C = \text{diag}(r_1 \ldots\ldots r_v) - NK^{-1}N'$

$rI_v - \dfrac{NN'}{k}$ { $r_1 = r_2 = r_3 = \ldots = r_v = r$ in a PBIB Design)

$$2\, I_4 - \frac{1}{2}\begin{bmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1/2 & 0 & -1/2 \\ -1/2 & 1 & -1/2 & 0 \\ 0 & -1/2 & 1 & -1/2 \\ -1/2 & 0 & -1/2 & 1 \end{bmatrix}$$

$$\therefore 2C = \left[\begin{array}{cc|cc} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ \hline 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{array}\right] = \begin{bmatrix} 3I_2 - E_{22} & I_2 - E_{22} \\ I_2 - E_{22} & 3I_2 - E_{22} \end{bmatrix}$$

$$\therefore \theta_1 = \frac{3}{2} + \frac{1}{2} = 2, \qquad \theta_2 = \frac{3}{2} - \frac{1}{2} = 1$$

Example:

$$\begin{bmatrix} 1 & 4 & 2 & 5 \\ 2 & 5 & 3 & 6 \\ 3 & 6 & 1 & 4 \end{bmatrix} \qquad \begin{array}{ccccccc} 1 & \rightarrow & 4 & 2 & 3 & 5 & 6 \\ 2 & \rightarrow & 5 & 1 & 3 & 4 & 6 \\ 3 & \rightarrow & 6 & 1 & 2 & 4 & 5 \\ 4 & \rightarrow & 1 & 2 & 3 & 5 & 6 \\ 5 & \rightarrow & 2 & 1 & 3 & 4 & 6 \\ 6 & \rightarrow & 3 & 1 & 2 & 4 & 5 \end{array}$$

here, v = 6, b = 3, r = 2, k = 4, λ_1 = 2.

n_2 = 4, n_1 = 1, n = n_1 +1 = 2

$$m = \frac{n_2}{n} + 1 = \frac{4}{2} + 1 = 3.$$

mn = 2(3) = 6.

v = 6 = mn and r -λ_1 = 2-2=0

∴ This is a singular group divisible design.

Incidence Matrix N can be written as

		Block		
		1	2	3
	1	1	0	1
t	2	1	1	0
r	3	0	1	1
e	4	1	0	1
a	5	1	1	0
t	6	0	1	1

$$N' = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}_{3\times 6}$$

$$
= \begin{bmatrix} 2 & 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 & 2 \end{bmatrix} = NN'
$$

$\therefore$ NN' = $rB_0 + \lambda_1 B_1 + \lambda_2 B_2$

$$
\text{now , } C = rI_v - \frac{NN'}{k} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 2 & 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 & 2 \end{bmatrix}
$$

$$\therefore NN' = \begin{bmatrix} 2 & 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & 1 & 2 \end{bmatrix}_{6\times 6}$$

now , $rB_0 + \lambda_1 B_1 + \lambda_2 B_2$

$$= 2\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$r \qquad\qquad B_0 \qquad\qquad\qquad B_1 \qquad\qquad\qquad B_2$$

$$
=\begin{bmatrix}
6/4 & -1/4 & -1/4 & -2/4 & -1/4 & -1/4 \\
-1/4 & 6/4 & -1/4 & -1/4 & -2/4 & -1/4 \\
-1/4 & -1/4 & 6/4 & -1/4 & -1/4 & -2/4 \\
-2/4 & -1/4 & -1/4 & 6/4 & -1/4 & -1/4 \\
-1/4 & -2/4 & -1/4 & -1/4 & 6/4 & -1/4 \\
-1/4 & -1/4 & -2/4 & -1/4 & -1/4 & 6/4
\end{bmatrix}
$$

$$
\therefore 4C=\left[\begin{array}{ccc:ccc}
6 & -1 & -1 & -2 & -1 & -1 \\
-1 & 6 & -1 & -1 & -2 & -1 \\
-1 & -1 & 6 & -1 & -1 & -2 \\
\hdashline
-2 & -1 & -1 & 6 & -1 & -1 \\
-1 & -2 & -1 & -1 & 6 & -1 \\
-1 & -1 & -2 & -1 & -1 & 6
\end{array}\right]
=\begin{bmatrix}
7I_3-E_{33} & -I_3-E_{33} \\
-I_3-E_{33} & 7I_3-E_{33}
\end{bmatrix}
$$

Here $\theta_1 = 8/4 = 2$ and $\theta_2 = (8-1)/4 = 7/4$

Intrablock Analysis of a PBIBD :

Let $S_i\,(t_\alpha)$ = sum of those treatments which are i^{th}

associate of treatment t_α

Let $S_i\,(t_\alpha) = \sum_{u=1}^{v} b^i_{\alpha u} t_u = \alpha^{th}$ element of $B_i\,\underline{t}$

and $S_j\,S_i\,(t_\alpha)$ = sum of those treatments which are j^{th} associate of i^{th} associates of treatment t_α .

Lemma: Show that that for m-associate classes PBIBD,

$$S_j\,S_i\,(t_\alpha) = \sum_{u=0}^{m} b^k_{ji} S_u(t_\alpha)$$

Proof: $B_j B_i \underline{t} = (B_j B_i)\underline{t} = \sum_{u=0}^{m} (p_{ji}^{u} B_u)\,\underline{t}$ (if we multiply two associate treatments i^{th} and j^{th} then product will give $\sum p_{ji}$ of the u associates u = 1( ) m.

$$= \sum_{u=0}^{m} p_{ji}^{u} (B_u\ \underline{t})$$

hence α^{th} element of $B_j\ B_i\ \underline{t}$ is $= \sum_{u=0}^{m} p_{ji}^{u}\ S_u(\,t_\alpha)$ (1)

also $B_j\ B_i\ \underline{t} = B_j\ (B_i\ \underline{t}\ )$

$= B_j\ \{\ S_i\ (t_1)\ \ S_i\ (t_2) \ldots S_i\ (t_\alpha) \ldots S_i\ (t_v)\ \ \}$

hence α^{th} element of $B_j\ B_i\ \underline{t}$ is $\sum_{w=1}^{v} b_{\alpha w}^{v}\ S_i(\,t_w)$

= sum of all those treats which are j^{th} associate of treatment t_α (for the i^{th} associate group)

$= S_j\ S_i\ (t_\alpha)$ (2)

Hence from (1) and (2), $S_j S_i(t_\alpha) = \sum_{u=0}^{m} p_{ji}^{u} S_u(t_\alpha)$

Intra block Analysis of two associate PBIBD:

The reduced normal equation for the estimates of treatment effects are $\underline{Q} = C\hat{\underline{t}}$

Where $C = R - NK^{-1}N'$

Hence for 2-associate class PBIBD we get –

$$= rI_v - k^{-1}NN' = rI_v - k^{-1}\sum_{i=0}^{m=2}\lambda_i B_i = rI_v - k^{-1}[\lambda_0 B_0 + \sum_{i=1}^{2}\lambda_i B_i]$$

$$C = rI_v - k^{-1}[rI_v + \sum_{i=1}^{2}\lambda_i B_i] = r(1-k^{-1})I_v - k^{-1}\sum_{i=1}^{2}\lambda_i B_i$$

$$\underline{Q} = C\hat{\underline{t}} \quad = [r(1-k^{-1})I_v - k^{-1}\sum_{i=1}^{2}\lambda_i B_i]\hat{\underline{t}}$$

$$= r(1-k^{-1})\hat{\underline{t}} - k^{-1}\sum_{i=1}^{2}\lambda_i B_i \hat{\underline{t}} = r(1-k^{-1})\hat{\underline{t}} - k^{-1}\sum_{i=1}^{2}\lambda_i S_i(\hat{\underline{t}})$$

$$\therefore \underline{Q}_s = r(1-k^{-1})\hat{t}_s - k^{-1}\sum_{i=1}^{2}\lambda_i S_i(\hat{\underline{t}}_s) \text{ for some treatment say, s}$$

$$\therefore \quad k\underline{Q}_s = r(k-1)\hat{t}_s - \sum_1^2 \lambda_i S_i(\hat{\underline{t}}_s)$$

$$= r(k-1)\hat{t}_s - \lambda_1 S_1(\hat{\underline{t}}_s) - \lambda_2 S_2(\hat{\underline{t}}_s) \qquad (1)$$

we know that , for testing $\mu_0 : \underline{t} = \underline{0}$, $E_{1v}\,\underline{t} = 0$

$$E_{1v}\underline{t} = 0 \qquad \Rightarrow S_0(t_s) + S_1(t_s) + S_2(t_s) = 0$$

$$\therefore\ S_2(t_s) = -t_s - S_1(t_s)$$

Substituting $S_2(t_s)$ in (1) we get,

$$\therefore \quad k\underline{Q}_s = r(k-1)\hat{t}_s - \lambda_1 S_1(\hat{t}_s) - \lambda_2[-\hat{t}_s - S_1(\hat{\underline{t}}_s)]$$

$$= [r(k-1) + \lambda_2]\hat{t}_s + (\lambda_2 - \lambda_1)S_1(\hat{\underline{t}}_s)$$

$$= A\hat{t}_s + BS_1(\hat{\underline{t}}_s) \qquad (2)$$

where , A= $r(k-1) + \lambda_2$ and B = $\lambda_2 - \lambda_1$,

If we put value of $S_i(t_s) = -t_s - S_2(t_s)$ in (1) then

$$\therefore \quad k\underline{Q}_s = [r(k-1) + \lambda_1]t_s + (\lambda_1 - \lambda_2)S_2(\underline{t}_s)]$$

$$= A\hat{t}_s + BS_2(\hat{\underline{t}}_s)$$

where, A= $r(k-1)+\lambda_1$ and B = $\lambda_1 - \lambda_2$,

now summing over (2) for the treatments which are first associate of treatment $\hat{t}_s$ we get

$$S_1(k\underline{Q}_s) = AS_1(\hat{t}_s) + BS_1(S_1(\hat{\underline{t}}_s)) \qquad (3)$$

$$\because S_1S_1(\hat{\underline{t}}_s) = \sum_{u=0}^{2} p_{11}^{u} S_u(\hat{t}_s)$$

$$= p_{11}^{0} S_0(\hat{t}_s) + p_{11}^{1} S_1(\hat{t}_s) + p_{11}^{2} S_2(\hat{t}_s)$$

$$= n_1 \hat{t}_s + p_{11}^{1} S_1(\hat{t}_s) + p_{11}^{2} [-\hat{t}_s - S_1(\hat{t}_s)]$$

$$= (n_1 - p_{11}^{2}) \hat{t}_s + (p_{11}^{1} - p_{11}^{2}) S_1(\hat{t}_s)$$

$$= p_{12}^{2} \hat{t}_s + (p_{11}^{1} - p_{11}^{2}) S_1(\hat{t}_s) \quad \because p_{12}^{2} = n_1 - p_{11}^{2}\,]$$

hence (3) reduce to

$$S_1(k\underline{Q}_s) = AS_1(\hat{t}_s) + B[p_{12}^{2}\hat{t}_s + (p_{11}^{1} - p_{11}^{2})S_1(\hat{\underline{t}}_s)]$$

$$= Bp_{12}^{2}(\hat{t}_{s}) + [A + B(p_{11}^{1} - p_{11}^{2})]S_{1}(\underline{\hat{t}}_{s})]$$

$$= C\hat{t}_{s} + DS_{1}(\underline{\hat{t}}_{s})] \qquad (4)$$

where C = Bp_{12}^{2} and D =[$A + B(p_{11}^{1} - p_{11}^{2})$]

$$\mathrm{DkQ_s} = \mathrm{AD}\,\hat{\mathrm{t}}_{\mathrm{s}} + \mathrm{BDS_1}(\hat{\mathrm{t}}_{\mathrm{s}})$$

$$\mathrm{BS_1(kQ_s)} = \mathrm{BC}\,\hat{\mathrm{t}}_{\mathrm{s}} + \mathrm{BDS_1}(\hat{\mathrm{t}}_{\mathrm{s}})$$

$$\mathrm{DkQ_s} - \mathrm{BS_1(kQ_s)} = (\mathrm{AD} - \mathrm{BC})\,\hat{\mathrm{t}}_{\mathrm{s}}$$

$$= \mathrm{X}.\hat{\mathrm{t}}_{\mathrm{s}} \qquad \text{Where } \mathrm{X} = \mathrm{AD} - \mathrm{BC}$$

$$\therefore \hat{\mathrm{t}}_{\mathrm{s}} = \frac{1}{\mathrm{X}}[\mathrm{DkQ_s} - \mathrm{BS_1(kQ_s)}]$$

Adjusted Treatment S.S. $= \underline{\hat{\mathrm{t}}}'\,\underline{\mathrm{Q}} = = \sum_{\mathrm{s}=1}^{\mathrm{v}} \hat{\mathrm{t}}_{\mathrm{s}}\mathrm{Q_s}$

$$= \sum_{\mathrm{s}=1}^{\mathrm{v}} \frac{1}{\mathrm{X}}[\mathrm{DkQ_s} - \mathrm{BS_1(kQ_s)}]\mathrm{Q_s}$$

$$= \frac{1}{X}\left[\sum_{s=1}^{v}(DkQ_s^2 - BS_1(kQ_s)]Q_s\right]$$

$$= \frac{1}{X}\left[\sum_{s=1}^{v}(\frac{D}{k}(kQ_s)^2 - \frac{B}{k}S_1(kQ_s)(kQ_s))\right]$$

$$= \frac{D}{kX}\sum_{s=1}^{v}(kQ_s)^2 - \frac{B}{kX}\sum_{s=1}^{v}(kQ_s)(S_1(kQ_s))$$

$$= \frac{1}{kX}\left[D\sum_{s=1}^{v}(kQ_s)^2 - B\sum_{s=1}^{v}(kQ_s)S_1(kQ_s)\right] \qquad (5)$$

The other quantities are to be obtained as usual.

ANOVA TABLE

source	d.f.	S.S.	M.S.S. MSS	S.S.	d.f.	source
Block (unadj.)	b-1	$\frac{1}{k}(\sum B_j^2) - C.F.$	SB/b-1	+	b-1	Block(adj.)
Treat(adj)	v-1	(S)	(S)/v-1	$\frac{1}{v}\sum T_i^2 - C.F.$	v-1	treat(unadj)
Error	(bk-v-b+1)	SSE	SSE/df(E)	+		Error
Total	bk - 1	$\sum y^2 - CT$		$\sum y^2 - CT$	bk-1	Total

ANALYSIS OF TWO WAY DESIGN

In each of block design the treatment are selected randomly which called one way block design, i.e. we remove the heterogeneity of data in the one direction. Now one is interested to remove it in to two direction so we need blocking in two ways, i.e., treatments are selected randomly in row and then treatments are selected randomly in columns which we called row – column design or two way heterogeneity of design or simply two way design .

The model of two way design is given by:

$$y_{jk}^{(i)} = \mu + \alpha_j + \beta_k + \tau_i + e_{jk}^{(i)} \qquad (1)$$

when $y_{jk}^{(i)}$ is yield due to j^{th} row and k^{th} column for i^{th} treatment.

μ = general mean ; α_j = effect of j^{th} row

β_k = effect of k^{th} column ; τ_i = effect of i^{th} treatment

$e_{jk}^{(i)}$ = random error in j^{th} row and k^{th} column for i^{th} treatment .

Let there are n blocks arranged in u row and u' column. l_{ij} is a number of times v treatments are in u rows similarly, m_{ik} denotes the number of times v treatments are randomly allocated in u' column. Each treatment is replicated r_i times and each block contains k treatments.

i = 1,2,…v; j = 1,2,….u; k = 1,2,….u'

$$l_{ij} = \begin{array}{c} \\ \\ \\ \\ \\ Total \end{array}\begin{array}{c} \begin{bmatrix} l_{11} & l_{12} & \cdots & l_{1u} \\ l_{21} & l_{22} & \cdots & l_{2u} \\ \vdots & \vdots & \cdots & \vdots \\ l_{v1} & l_{v2} & \cdots & l_{vu} \end{bmatrix} \\ \begin{matrix} u' & u' & \cdots & u' \end{matrix} \end{array}\begin{array}{c} r_1 \\ r_2 \\ \vdots \\ r_v \\ \\ \end{array} \qquad \therefore u' = \sum_i l_{ij}$$

and

$$m_{ik} = \begin{array}{cc} \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1u'} \\ m_{21} & m_{22} & \cdots & m_{2u'} \\ \cdots & \cdots & \cdots & \cdots \\ m_{v1} & m_{v2} & \cdots & m_{vu'} \end{bmatrix} & \begin{array}{c} r_1 \\ r_2 \\ \vdots \\ r_v \end{array} \\ \begin{array}{cccc} u & u & \cdots & u \end{array} & \end{array}$$

Assumptions:

- The model is linear.
- The model is fixed effect.
- The model is additive.
- The model is homoschadastic.

$$e_{jk}^{(i)} \sim N(0, \sigma^2) \;\; And \quad E[y_{jk}^{(i)}] = \mu + \alpha_j + \beta_k + \tau_1 \;, \mathrm{var}(y_{jk}^{(i)}) = \sigma^2$$

here $\mu, \alpha_j, \beta_k, \tau_i$ are unknown parameters which are to be estimated.

Let R $= (R_{1,\ldots,}R_u)$; C $= (C_{1,\ldots..,}C_{u'})$

$$T = (T_1,\ldots,T_v)'$$

R_j is j^{th} row, C_k is k^{th} column and T_i is i^{th} treatment effects.

Now , $E = \sum_i \sum_j \sum_k e_{ijk}^{(i)2} = \sum_i \sum_j \sum_k (y_{jk}^{(i)} - \mu - \alpha_j - \beta_k - \tau_i)^2$ (2)

μ, α_j, β_k *and* τ_i are to be estimated from (2) using least square estimation.

$$\frac{dE}{d\mu} = 2 \sum_i \sum_j \sum_k (y_{jk}^{(i)} - \mu - \alpha_j - \beta_k - \tau_i)(-1) = 0$$

$$\therefore \sum_i \sum_j \sum_k y_{jk}^{(i)} = \sum_i \sum_j \sum_k \mu + \sum_{ijk} \alpha_j + \sum_{ijk} \beta_k + \sum_{ijk} \tau_i$$

$$G = uu'\mu + u'\sum_j \alpha_j + u\sum_k \beta_k + \sum_i uu'\tau_i$$

$$= uu'\mu + u'\sum_j \alpha_j + u\sum_k \beta_k + \sum_i r_i \tau_i \qquad (3)$$

$$(\because \quad uu' = \tau r_i = n)$$

$$\frac{dE}{d\alpha_j} = 2\sum_i\sum_k (y_{ik}^{(i)} - \mu - \alpha_j - \beta_k - \tau_i)(-1) = 0$$

$$\Rightarrow \sum_k y_{jk}^{(i)} = \sum_k \mu + \sum_k \alpha_j + \sum_k \beta_k + \sum_k \tau_i$$

$$\Rightarrow R_j = u'\mu + \mu'\alpha_j + \sum_k \beta_k + u'\tau_i$$

$$\Rightarrow R_j = u'\mu + u'\alpha_j + \sum_k \beta_k + \sum_i l_{ij}\tau_i \qquad (4)$$

$$\frac{dE}{d\beta_k} = 2\sum_j (y_{jk}^{(i)} - \mu - \alpha_j - \beta_k - \tau_i)(-1) = 0$$

$$\Rightarrow \sum_j y_{jk}^{(i)} = \sum_j \mu + \sum_j \alpha_j + \sum_j \beta_k + \sum_j \tau_i$$

$$C_k = u\mu + \sum_j \alpha_j + u\beta_k + u\tau_i$$

$$= u\mu + \sum_j \alpha_j + u\beta_k + \sum_i m_{ik}\tau_i \qquad (5)$$

$$\frac{dE}{d\tau_i} = 2\sum_j \sum_k (y_{jk}^{(i)} - \mu - \alpha_j - \beta_k - \tau_i)(-1) = 0$$

$$\Rightarrow \sum_j \sum_k y_{jk}^{(i)} = \sum_j \sum_k \mu + \sum_{jk} \alpha_j + \sum_{jk} \beta_k + \sum_{jk} \tau_i$$

$$\Rightarrow T_i = uu'\mu + u'\sum_j \alpha_j + u\sum_k \beta_k + uu'\tau_i$$

$$= \sum_i r_i \mu + \sum_i l_{ij} \sum_j \alpha_j + \sum_i m_{ik} \sum_k \beta_k + \sum_i r_i \tau_i$$

$$= r_i \mu + \sum_j l_{ij} \alpha_j + \sum_k m_{ik} \beta_k + r_i \tau_i \qquad (6)$$

Now , we have 1+u+u'+v normal equations. This normal equations are dependent because if we sum over all j of (4), sum over all k for (5) and sum over all v for (6), we reach at (3). Hence the unknown parameters μ, α_j , β_k , ι_i can not be estimated.

To estimate this parameters one has to put some restrictions. Let as convert this normal equation (3), (4), (5), (6) in the form of matrix.

$$\begin{bmatrix} G \\ \underline{R} \\ \underline{C} \\ \underline{T} \end{bmatrix} = \begin{bmatrix} uu' & u'E_{1u} & uE_{1u} & \underline{r}' \\ u'E_{u1} & u'I_{u} & Euu' & L' \\ u'E_{u'1} & Eu'u & uIu' & M' \\ \underline{r}' & L & M & diag.(r_1 \ldots r_v) \end{bmatrix} \begin{bmatrix} \mu \\ \underline{\alpha} \\ \underline{\beta} \\ \underline{\tau} \end{bmatrix} \qquad (7)$$

The normal equation (7) can be expressed in the form $A'\underline{y} = A'A\underline{\hat{\theta}}$

We have $Var(\underline{y}) = \sigma^2 In.$

$Var\,(A'\underline{y}) = A'Var\,(\underline{y})A = A'\sigma^2 I_n A = \sigma^2 A'A$

$$\therefore Var\begin{bmatrix} G \\ \underline{R} \\ \underline{C} \\ \underline{T} \end{bmatrix} = \sigma^2 \begin{bmatrix} uu' & u'E_{1u} & uE_{1u'} & \underline{r}' \\ u'E_{u1} & u'I_u & E_{uu'} & L' \\ uE_{u'1} & Eu'u & uIu' & M' \\ \underline{r}' & L & M & diag.(r_1 \ldots . r_v) \end{bmatrix}$$

To test the parameters $H_0 \therefore \underline{t} = 0$

We assume that $E_{1u}\,\underline{\alpha} = 0$, $E_{1u'}\,\underline{\beta} = 0$, $E_{1v}\underline{t} = 0$

From (7) we get ,

$$G = uu'\hat{\mu} + u'E_{1u}\hat{\underline{\alpha}} + uE_{1u'}\hat{\underline{\beta}} + \underline{r}'\hat{\underline{t}}$$
$$= uu'\hat{\mu} + \underline{r}'\hat{\underline{t}}$$

$$\therefore \hat{\mu} = \frac{1}{uu'}[G - \underline{r}'\hat{\underline{t}}] \qquad (8)$$

$$\underline{R} = u'E_{u1}\hat{\mu} + u'I_u\hat{\underline{\alpha}} + E_{uu'}\hat{\underline{\beta}} + L'\hat{\underline{t}}$$
$$= \hat{\mu}u'E_{u1} + u'\hat{\underline{\alpha}} + L'\hat{\underline{t}}$$

$$\therefore \hat{\underline{\alpha}} = \frac{1}{u'}[\underline{R} - \hat{\mu}u'E_{u1} - L'\hat{\underline{t}}] \qquad (10)$$

$$\underline{C} = uE_{u'1}\hat{\mu} + E_{u'u}\,\underline{\hat{\alpha}} + uI_{u'}\,\underline{\hat{\beta}} + M'\,\underline{\hat{t}}$$

$$= \hat{\mu}uE_{u'1} + u\underline{\hat{\beta}} + M'\,\underline{\hat{t}} \qquad (11)$$

$$\underline{\hat{\beta}} = \frac{1}{u}[\underline{C} - \hat{\mu}uE_{u'1} - M'\,\underline{\hat{t}}]$$

$$\underline{T} = \underline{r}\hat{\mu} + L\underline{\hat{\alpha}} + M\underline{\hat{\beta}} + \mathrm{diag}.(r_1,\ldots,r_v)\,\hat{t}.$$

$$\therefore \underline{T} = \underline{r}\hat{\mu} + \frac{L\underline{R}}{u'} - \hat{\mu}LE_{u1} - \frac{LL'\underline{\hat{t}}}{u'} + \frac{M\underline{C}}{u} - \hat{\mu}ME_{u'1} -$$

$$\frac{MM'\underline{\hat{t}}}{u} + diag.(r_1,\ldots,r_v)\,\underline{\hat{t}}$$

$$= \frac{L\underline{R}}{u'} + \frac{M\underline{C}}{u} - \underline{r}\hat{\mu} + \mathrm{diag}.(r_1,\ldots,r_v)\,\underline{\hat{t}} - \frac{LL'\underline{\hat{t}}}{u'} - \frac{MM'\underline{\hat{t}}}{u}$$

$$= \frac{L\underline{R}}{u'} + \frac{M\underline{C}}{u} - \frac{rG}{uu'} + \frac{rr'\hat{t}}{uu'} + \mathrm{diag}(r_1,\ldots,r_v)\,\underline{\hat{t}} - \frac{LL'\underline{\hat{t}}}{u'} - \frac{MM'\underline{\hat{t}}}{u}$$

$$\underline{T} - \frac{L\underline{R}}{u'} - \frac{M\underline{C}}{u} + \frac{rG}{uu'} = [\mathrm{diag}(r_1,\ldots,r_v) - \frac{LL'}{u'} - \frac{MM'}{u} + \frac{\underline{r}\,\underline{r}'}{uu'}]\underline{\hat{t}}$$

$$\underline{Q} = F\underline{\hat{t}} \quad \text{, where } \underline{Q} = \underline{T} - \frac{LR}{u'}. - \frac{MC}{u} + \frac{\underline{r}G}{uu'}$$

and F = diag $(r_1,\ldots,r_v) - \dfrac{LL'}{u'} - \dfrac{MM'}{u} + \dfrac{\underline{r}\underline{r}'}{uu'}$

Evidently we get $E_{1v}\ \underline{Q} = 0$

$E(\underline{Q}) = F\underline{t}$

$Var(\underline{Q}) = \sigma^2 F$

$E_{1v}\ F = \underline{0}'$ and $FE_{v1} = \underline{0}$

$\therefore$ Rank (F) = v-1

when Rank (F) = v-1 all treatment contrasts are estimable

Test : $\mu_0 : \underline{t} = \underline{0}$

We have,

$$\hat{\mu} = \frac{G}{uu'} - \frac{\underline{r}'\hat{\underline{t}}}{uu'} = \frac{1}{uu'}[G - \underline{r}'\hat{\underline{t}}]$$

$$\hat{\underline{\alpha}} = \frac{\underline{\underline{R}}}{u'} - \hat{\mu}E_{ui} - \frac{L'\hat{\underline{t}}}{u'} = \frac{1}{u'}[\underline{R} - \hat{\mu}\,u'E_{u1} - L'\hat{\underline{t}}]$$

$$\hat{\underline{\beta}} = \frac{\underline{\underline{C}}}{u} - \hat{\mu}E_{u'1} - \frac{M'\underline{t}}{u} = \frac{1}{u}[\underline{C} - \hat{\mu}uE_{u'1} - M'\hat{\underline{t}}]$$

$$SSR\,(\mu', \underline{\hat{\alpha}}, \underline{\hat{\beta}}, \underline{\hat{t}}\,) = \underline{\hat{\theta}}'A'\underline{y}$$

$$= \mu'G + \underline{\hat{\alpha}}\,\underline{R} + \underline{\hat{\beta}}\,\underline{C} + \underline{\hat{t}}\;\underline{T}$$

$$= \hat{\mu}G + \frac{\underline{R}'\underline{R}}{u'} - \hat{\mu}E_{1u}\underline{R} - \frac{\hat{t}'L\underline{R}}{u'} + \frac{\underline{C}'\underline{C}}{u} - \hat{\mu}E_{1u'}\underline{C} - \frac{\hat{t}'M\underline{C}}{u} + \underline{\hat{t}}'\underline{T}$$

$$= \frac{\underline{R}'\underline{R}}{u'} - \hat{\mu}G + \underline{\hat{t}}'\underline{T} - \frac{\hat{t}'L\underline{R}}{u'} + \frac{\hat{t}'M\underline{C}}{u} + \frac{\underline{C}'\underline{C}}{u}$$

$$= \frac{\underline{\underline{R}}'\underline{\underline{R}}}{u'} + \frac{\underline{\underline{C}}'\underline{\underline{C}}}{u} - \frac{G^2}{uu'} + \hat{\underline{t}}'\underline{T} + \frac{\underline{r}'\hat{\underline{t}}G}{uu'} - \frac{\hat{t}'L\underline{\underline{R}}}{u'} - \frac{\hat{t}'M\underline{\underline{C}}}{u}$$

$$= \frac{\underline{\underline{R}}'\underline{\underline{R}}}{u'} + \frac{\underline{\underline{C}}'\underline{\underline{C}}}{u} - \frac{G^2}{uu'} + \hat{\underline{t}}\left[\underline{T} - \frac{L\underline{\underline{R}}}{u'} - \frac{M\underline{\underline{C}}}{u} + \frac{rG}{uu'}\right]$$

$$= \frac{\underline{\underline{R}}'\underline{\underline{R}}}{u'} + \frac{\underline{\underline{C}}'\underline{\underline{C}}}{u} - \frac{G^2}{uu'} + \hat{\underline{t}}\left[\underline{T} - \frac{L\underline{\underline{R}}}{u'} - \frac{M\underline{\underline{C}}}{u} + \frac{rG}{uu'}\right]$$

$$= \frac{\underline{\underline{\mathrm{R}}}'\underline{\underline{\mathrm{R}}}}{\mathrm{u}'} + \frac{\underline{\underline{\mathrm{C}}}'\underline{\underline{\mathrm{C}}}}{\mathrm{u}} - \frac{\mathrm{G}^2}{\mathrm{uu}'} + \hat{\underline{\mathrm{t}}}\underline{\mathrm{Q}}$$

d.f. for $SSR\,(\mu', \hat{\underline{\alpha}}, \hat{\underline{\beta}}, \hat{\underline{t}}\,) = u + u' - 1 + Rank\,(F)$

$$= \underline{y}'\underline{y} - SSR\,(\mu', \hat{\underline{\alpha}}, \hat{\underline{\beta}}, \hat{\underline{t}}\,)$$

$$= \underline{\mathrm{y}}'\underline{\mathrm{y}} - \frac{\underline{\underline{\mathrm{R}}}'\underline{\underline{\mathrm{R}}}}{\mathrm{u}'} - \frac{\underline{\underline{\mathrm{C}}}'\underline{\underline{\mathrm{C}}}}{\mathrm{u}} + \frac{\mathrm{G}^2}{\mathrm{uu}'} + \hat{\underline{\mathrm{t}}}\underline{\mathrm{Q}}$$

$$= \left(\underline{\mathrm{y}}'\underline{\mathrm{y}} - \frac{\mathrm{G}^2}{\mathrm{uu}'}\right) - \left(\frac{\underline{\underline{\mathrm{R}}}'\underline{\underline{\mathrm{R}}}}{\mathrm{u}'} - \frac{\mathrm{G}^2}{\mathrm{uu}'}\right) - \left(\frac{\underline{\underline{\mathrm{C}}}'\underline{\underline{\mathrm{C}}}}{\mathrm{u}} - \frac{\mathrm{G}^2}{\mathrm{uu}'}\right) - \hat{\underline{\mathrm{t}}}\underline{\mathrm{Q}}$$

d.f. for SSE are $= uu' - [u + u' - 1 + Rank(F)]$

$$= (u'-1)(u-1) - Rank(F).$$

now under $:\mu_0 : \underline{t} = \underline{0}$, we get.

$$SSR(\mu_0) = \underline{\theta}^{*'} A' \underline{y}$$

$$= SSR\ (\mu^*, \underline{\alpha}^*, \underline{\beta}^*)$$

$$= \frac{\underline{R}'\underline{R}}{u'} + \frac{\underline{C}'\underline{C}}{u} - \frac{G^2}{uu'}$$

sum square due to $:\mu_0 : \underline{t} = \underline{0}$ is

$$= \text{SSR} - \text{SSR}\ (\mu_0)$$

$$= \underline{\hat{t}}\,\underline{Q}\,.$$

d.f. for SS due to $:\mu_0 : \underline{t} = \underline{0}$ are Rank(F) .

Hence the test for testing $:\mu_0 : \underline{t} = \underline{0}$ is

$$\mathrm{F} = \frac{\underline{\hat{\mathrm{t}}}'\,\underline{\mathrm{Q}} \big/ \mathrm{Rank}(\mathrm{F})}{\mathrm{SSE} \big/ [(\mathrm{u}-1)(\mathrm{u}'-1) - \mathrm{Rank}(\mathrm{F})]}$$

ANOVA TABLE : (For testing : $H_0 : \underline{t} = \underline{0}$)

Sources	d.f.	S.S.	M.S.S.	F_c
Rows(unadj.)	u-1	$\frac{1}{u'}\sum_j R_j^2 - CT$		
Column(unadj.)	u'-1	$\frac{1}{u'}\sum_k R_k^2 - CT$		
Treat(adj.)	Rank(F)	$\hat{\underline{t}}'\underline{Q}$	$\hat{\underline{t}}'\underline{Q}/R(F)$	$\frac{\hat{\underline{t}}'\underline{Q}/R(F)}{S}$
Error	a	b	b/a(=S)	
Total	uu'-1	$\sum y^2 - CT$		

On similar lines one can obtain test for testing (i) $H_0 : \underline{\alpha} = \underline{0}$
(ii) $H_0 : \underline{\beta} = \underline{0}$

Particular Designs :

Latin square designs

Definition: A LSD is an arrangement of v treatments in v^2 plots arranged in v rows and v column that every treatment occurs exactly once in each rows and each column.

Remarks: (i) These designs require equal number of treatments and replication .In this case number of rows equal to number of column equal to number of treatments.

(ii) A LSD with v treatments is a LSD of order v.

(iii) A LSD with symbols in first row and first column in natural order is called a regular LSD and a LSD with symbols in first row in natural order is called a semi-regular LSD.

Analysis: $H_0 : \underline{t} = \underline{0}$

$u = v = u'$ and $L = E_{vv} = M$.

$$\underline{Q} = \underline{T} - \frac{L\underline{R}}{u'} - \frac{M\underline{C}}{u} + \frac{\underline{r}G}{vu'}$$

$$= \underline{\mathrm{T}} - \frac{\mathrm{L}\underline{\mathrm{R}}}{\mathrm{u}'} - \frac{\mathrm{M}\underline{\mathrm{C}}}{\mathrm{u}} + \frac{\mathrm{L}\mathrm{E}_{\mathrm{u}1}\mathrm{G}}{\mathrm{vu}'}$$

$$= \underline{\mathrm{T}} - \frac{\mathrm{E}_{\mathrm{vv}}\underline{\mathrm{R}}}{\mathrm{v}} - \frac{\mathrm{E}_{\mathrm{vv}}\underline{\mathrm{C}}}{\mathrm{v}} + \frac{\mathrm{E}_{\mathrm{vv}}\mathrm{E}_{\mathrm{v}1}\mathrm{G}}{\mathrm{v}^2}$$

$$= \underline{\mathrm{T}} - \frac{\mathrm{G}\mathrm{E}_{\mathrm{v}1}}{\mathrm{v}} - \frac{\mathrm{G}\mathrm{E}_{\mathrm{v}1}}{\mathrm{v}} + \frac{\mathrm{G}\mathrm{E}_{\mathrm{v}1}}{\mathrm{v}}$$

$$= \underline{\mathrm{T}} - \frac{\mathrm{G}\mathrm{E}_{\mathrm{v}1}}{\mathrm{v}}$$

In LSD we have $r_1 = r_2 = \ldots = r_v = v$

$$\mathrm{F} = \mathrm{diag}\,(\mathrm{r}_1,\ldots,\mathrm{r}_\mathrm{v}) - \frac{\mathrm{LL}'}{\mathrm{u}'} - \frac{\mathrm{MM}'}{\mathrm{u}} + \frac{\mathrm{rr}'}{\mathrm{uu}'}$$

$$= \mathrm{vI}_\mathrm{v} - \frac{\mathrm{LL}'}{\mathrm{u}'} . - \frac{\mathrm{MM}'}{\mathrm{u}} + \frac{\mathrm{L}\mathrm{E}_{\mathrm{v}1}\mathrm{E}_{1\mathrm{v}}\mathrm{L}'}{\mathrm{uu}'}$$

$$= vI_v - \frac{vE_{vv}}{v} - \frac{vE_{vv}}{v} + E_{vv}E_{v1}E_{1v}E_{vv}/vv$$

$$= vI_v - E_{vv} \quad = \quad v\left(I_v - \tfrac{1}{v}E_{vv}\right)$$

$\therefore$ Rank (F) = v-1

$$Q = F\hat{\underline{t}}$$

$$= v\left(I_v - \tfrac{1}{v}E_{vv}\right)\hat{\underline{t}}$$

$$= v\hat{\underline{t}} - E_{vv}\hat{\underline{t}} \;\; = v\hat{\underline{t}}$$

$$\therefore \hat{\underline{t}} = (1/v)\ \underline{Q}$$

$$\therefore \text{ Adj. tr. S.S.} = \underline{t}'\underline{Q}$$

$$= (1/v)\underline{Q}'\,\underline{Q}$$

$$= \frac{1}{v}\left[\underline{T}' - \frac{GE_{1v}}{v}\right]\left[\underline{T} - \frac{GE_{v1}}{v}\right]$$

$$= \frac{1}{v}\left[\underline{T}'\underline{T} - \frac{G\underline{T}'E_{v1}}{v} - \frac{GE_{1v}\underline{T}}{v} + \frac{G^2v}{v^2}\right]$$

$$= \frac{1}{v}\left[\underline{T}'\underline{T} - \frac{GG}{v} - \frac{GG}{v} + \frac{G^2 v}{v^2} \right]$$

$$= \frac{1}{v}\left[\underline{T}'\underline{T} - \frac{G^2}{v} \right]$$

$$\text{Adj. tr. S.S.} = \frac{1}{v}\sum_{i=1}^{v} T_i^2 - G^2 / v^2$$

ANOVA TABLE : (For testing $: H_0 : \underline{t} = \underline{0}$)

Sources	d.f.	S.S.	M.S.S.	F_c
Rows	v-1	$\frac{1}{v}\sum_i R_i^2 - TC$		
Column	v-1	$\frac{1}{v}\sum_j C_j^2 - TC$		
Treats	v-1	$\frac{1}{v}\sum_i T_i^2 - TC$	MST	MST/MSE
Error	+	+	MSE	
Total	v^2-1	$\sum y^2 - CT$		

$\therefore$ The test for testing : $\mu_0 : \underline{t} = \underline{0}$ is $F_c = \dfrac{\hat{\underline{t}}'\underline{Q} \big/ (v-1)}{MSE}$

on similar line one can obtain test for sig. For testing (i) $H_0 : \underline{\alpha} = \underline{0}$

(ii) $H_0 : \underline{\beta} = \underline{0}$

Remarks : (i) We know that d.f. carried by SSE are (v-1)(v-2). Hence d.f. carried by SSE of a LS of order v = 2 are (v-1)(v-2) = 0. There fore for smaller number of treatments, d.f. for SSE of LSD are very few. Hence LSD`s are not suitable for smaller number of treatments.

(ii) The analysis of LSD become very much complicated when several plot yields are missing .

(iii) LSD`s less flexible.

CROSS OVER DESIGNS :-

We have seen that when number of treats are smaller , LSD`s are not suitable . In such situations Cross over designs are used. These designs are widely used in animal husbandry.

Cross Over Designs resemble with LSD`s from analysis point of view , these are nothing but P replications of v×v LS`s . Thus variation from P replications is also eliminated from error s.s.

Let there be v treatments arranged in v rows and vp columns such that there are p replications of v×v LS.

Hence we get.

$u = v, \quad u' = vp \; ; \quad l_{ij} = 1; \; i = 1,2,\ldots,v; \quad j = 1,2,\ldots..v.$

$\therefore L = E_{vv}$

$m_{ik} = 1, \quad i = 1,2,\ldots..v. \qquad k = 1,2,\ldots\ldots,vp \qquad \therefore M = E_{v(vp)}$

$$\underline{\theta} = \underline{T} - \frac{L\underline{\underline{R}}}{u'} - \frac{M\underline{\underline{C}}}{u} + \frac{\underline{\underline{R}}\,\underline{\underline{G}}}{uu'}$$

$$= \underline{\mathrm{T}} - \frac{\mathrm{L}\underline{\underline{\mathrm{R}}}}{\mathrm{u}'} - \frac{\mathrm{M}\underline{\underline{\mathrm{C}}}}{\mathrm{u}} + \frac{\mathrm{L}\mathrm{E}_{\mathrm{u1}}\mathrm{G}}{\mathrm{uu}'}$$

$$= \underline{\mathrm{T}} - \frac{\mathrm{E}_{\mathrm{vv}}\underline{\mathrm{R}}}{\mathrm{vp}} - \frac{\mathrm{E}_{\mathrm{v(vp)}}\underline{\mathrm{C}}}{\mathrm{v}} + \frac{\mathrm{E}_{\mathrm{vv}}\mathrm{E}_{\mathrm{u1}}\mathrm{G}}{\mathrm{v(vp)}}$$

$$= \underline{\mathrm{T}} - \frac{\mathrm{G}\mathrm{E}_{\mathrm{v1}}}{\mathrm{vp}} - \frac{\mathrm{G}\mathrm{E}_{\mathrm{v1}}\underline{\mathrm{C}}}{\mathrm{v}} + \frac{\mathrm{E}_{\mathrm{v1}}\mathrm{G}}{\mathrm{vp}}$$

$$= \underline{\mathrm{T}} - \frac{\mathrm{G}\mathrm{E}_{\mathrm{v1}}}{\mathrm{v}}$$

$$\mathrm{F} = \mathrm{diag.}\,(\mathrm{r}_1,\mathrm{r}_2\ldots\ldots\mathrm{r}_\mathrm{v}) - \frac{LL'}{u'} . - \frac{MM'}{u} + \frac{\underline{r}\,\underline{r}'}{uu'}$$

$$= vpI_v - \frac{E_{vv}E_{vv}}{vp} - \frac{E_{vvp}E_{vvp}}{v} + \frac{E_{vv}E_{vv}E_{vv}}{v(vp)}$$

$$\left[\begin{array}{l} \because \underline{r}\,\underline{r}' = LE_{v1}E_{1v}L' \\ \qquad\;\; = LE_{vv}L' \end{array}\right.$$

$$= vpI_v - \frac{E_{vv}}{p} - pE_{vv} + \frac{E_{vv}}{p}$$

$$= vpI_v - pE_{vv} \qquad = vp[I_v - \frac{1}{v}E_{vv}]$$

$\therefore$ Rank (F) = v-1

$$Q = F\underline{\hat{t}} = vp[I_v - \frac{1}{v}E_{vv}]\underline{\hat{t}}$$

$$= vpI_v\underline{\hat{t}} - pE_{vv}\underline{\hat{t}}$$

$$= vp\underline{\hat{t}}$$

$$\therefore \underline{\hat{t}} = \frac{\underline{Q}}{vp}$$

$$\text{Adj. tr. S.S.} = \underline{\hat{t}'}\,\underline{Q} \qquad = \qquad \frac{1}{v}p\underline{Q}'\,\underline{Q}$$

$$= \frac{1}{vp}[\underline{T}' - GE_{v}/v]\,[\underline{T} - GE_{v1}/v]$$

$$= \frac{1}{vp}\left[\underline{T}'\underline{T} - \frac{GT'E_{v1}}{v} - \frac{GE_{1v}\underline{T}}{v} + \frac{G^2E_{1v}E_{v1}}{v^2}\right]$$

$$= \frac{1}{vp}\left[\underline{T}'\underline{T} - \frac{G^2}{v} - \frac{G^2}{v} + \frac{G^2}{v}\right]$$

$$= \frac{1}{vp}\left[\underline{T}'\underline{T} - \frac{G^2}{v}\right]$$

$$= \frac{1}{vp}\left[\sum_{i=1}^{v} T_i^2 - \frac{G^2}{v(vp)}\right]$$

ANOVA TABLE : (For testing : $\mu_0 : \underline{t} = \underline{0}$)

Sources	d.f.	S.S.	M.S.S.
Rows	v-1	$\frac{1}{vp}\left(\sum_i R_i^2\right) - (TC)$	
Column	vp-1	$\frac{1}{v}\sum_j C_j^2 - TC$	
Treats	v-1	$\frac{1}{v}\sum T_i^2 - TC$	
Error	(v-1)(vp-2)	+	MSE
Total	v^2p-1	$\sum y^2 - CT$	

hence fortesting $H_0 : \underline{t} = \underline{0}$

$$Fc = \frac{\left[\frac{1}{vp}\sum_{i=1}^{v} T_i^2 - CT\right] \Big/ (\theta - 1)}{MSE}$$