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Aloke Dey

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Indian Statistical Institute, New Delhi, India

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Preface

Block designs have applications in almost all areas of human investigation including agriculture, biology, engineering, medicine, physical and chemical sciences and industrial experimentation. The most primitive of the block designs is the randomized (complete) block design. However, in many practical situations, adoption of a complete block design is not appropriate and in some cases, not at all feasible. This fact prompted the development of various kinds of incomplete block designs, which in turn have been used extensively for experiments in a variety of fields. Moreover, these designs opened up many challenging problems in combinatorial mathematics. In view of the importance of block designs both from a theoretical and practical perspective, the author published a book *Theory of Block Designs* in 1986, which was well received in academic circles. However, the book went out of print around 1992. The author initially toyed with the idea of bringing out a second edition of the book, incorporating only minor additions/changes. While attempting to do so, however, it was realized that during the intervening period, the subject has grown considerably and the emphasis on certain topics has shifted. The author therefore decided to write the present book which, while retaining some of the flavor of the earlier book, is substantially different from it in both coverage and presentation.

The literature on incomplete block designs is vast and it is near impossible to cover each and every development in incomplete block designs in a single book of reasonable length. In this book, an attempt has been made to cover all the developments in this area which in the author's perception are the major ones. Since the classical incomplete block designs like the balanced incomplete block and partially balanced incomplete block designs are still found useful in several applications and newer applications of these, e.g., in visual cryptography, have been found, such designs have been covered at some length. Some of the more recent developments in incomplete block designs for special types of experiments, like biological assays and diallel crosses have also been discussed. Important results on the optimality aspects of various incomplete block designs are also reviewed.

The book is organized into six chapters followed by an appendix. A brief description of the chapter contents appears in Chapter 1. The appendix covers some essentials on linear algebra, linear statistical models, finite fields and finite geometries. There are a large number of exercises at the end of Chapters 2–6 and there is a fairly exhaustive bibliography.

Throughout, results in matrix theory are used extensively and thus, a background in basic linear algebra and theory of matrices will be helpful in reading the book. Familiarity with the general area experimental designs and linear statistical models at an advanced undergraduate level is also assumed.

The book can be used in a variety of ways. The material in Chapter 2, the first four sections of Chapter 3 and Sections 4.1–4.4 and 4.6 of Chapter 4 can provide a solid foundation of the theory of incomplete block designs for a master's level course. The material in Sections 3.5–3.7, 4.7–4.11 and that in Chapters 5 and 6 may be used as a basis of a more advanced course. While the book is addressed to an audience whose primary interest is in the theory and applications of statistical design of experiments, portions of the book can also be used for a course in combinatorial designs for mathematicians. The material in Chapters 5 and 6 may be found useful for research students and consulting statisticians.

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Aloke Dey

February, 2010
New Delhi

Chapter 1

Introduction

1.1 Prologue

The modern foundations of design of experiments were laid by R. A. Fisher during the early part of the 20th century. Since then, this area has seen a phenomenal growth. Design of experiments has for long been an integral part of almost all scientific investigations and continues to be so. It has therefore played a fundamental role in statistical practice and research. Statistical training also has always emphasized the role of design of experiments in extracting correct information and making valid inference on the underlying problem and thus, design of experiments is an essential component of most statistics curricula.

While designing an experiment, the principles of randomization, replication and local control are of vital importance. These principles were first enunciated by Fisher while planning agricultural experiments. It was observed by Fisher that a completely random allocation of treatments to the experimental units, leading to a completely randomized design, eliminates bias in assessing treatment differences.

In certain experimental situations, there may be systematic variations present among the experimental units. For example, in a field experiment, the experimental units are typically plots of land. In such an experiment, there may be a fertility gradient present such that plots on the same fertility level are more homogeneous than those which are at different fertility levels. In experiments with piglets as experimental units, it is very plausible that piglets belonging to the same litter are genetically closer to each other (being born to the same pair of parents) than those belonging to different litters. Similarly, in experiments with livestock, different breeds (or, different ages) might be involved and animals belonging to the same breed are expected to be more alike than the ones belonging to different breeds. In the context of clinical trials with patients forming the experimental units, the trial may be conducted at different centers (mainly to get enough number of observations) and patients from the same center may be more alike than those from different centers due to differences in treatment practices and/or management procedures followed at different centers.

The above examples, which are merely illustrative and by no means exhaustive, demonstrate that in many situations there is a systematic variation among the experimental units. In such situations, use of a completely randomized design is not appropriate. Rather, one should take advantage of the *a priori* information about this systematic variation while designing the experiment in the sense that this information should be used while designing to eliminate the effect of such variability. The impact of this effort will be reflected in a reduced error, thereby increasing the sensitivity of the experiment. The above considerations led to the notion of local control or *blocking*. The groups of relatively homogeneous experimental units are called *blocks*. When the blocking is done according to one attribute, we get a *block design*. In a block design, the treatments are applied randomly to the experimental units within a block, the randomized allocation of treatments to experimental units within a block being done independently in each block.

The simplest among the block designs is the randomized complete block design. In such a design, each block is required to have as many experimental units as the number of treatments, i.e., the block size is equal to the number of treatments. However, it is not always possible to adopt a randomized complete block design in every experimental situation. Firstly, if one assumes that the intra-block variance is directly dependent on the block size, then adoption of a design with blocks of small sizes is preferable over one which has large block sizes. This restricts the use of randomized complete block designs in situations where the number of treatments is large. For example, in agronomic experiments, the experimenter generally chooses a block of size 10-12 and if this is accepted, then one cannot adopt a randomized complete block design in situations where say 20 treatments are to be compared. Furthermore, in many experimental situations, the block size is determined by the nature of the experiment. For example, with some experiments in psychology, it is quite common to consider the two members of a twin pair as experimental units of a block. In that case, clearly a randomized complete block design cannot be prescribed if the number of treatments is larger than two. Similarly, it is reasonable to take litter-mates (of say mice) as units of a block and litter size may not be adequate to accommodate all the treatments under test.

The few examples considered above clearly show that in many situations, one cannot adopt a randomized complete block design and thus, there is a need to look for designs where not all treatments appear in each block. Such designs are termed as *incomplete block designs*. The present book deals with block designs in general and their analysis, with special emphasis on certain important classes of incomplete block designs. The terms block design and incomplete block design are used interchangeably whenever there is no scope for confusion.

A reasonable amount of familiarity with basic notions of vector spaces and the algebra of matrices is assumed throughout and one may refer e.g., to Bapat (2000) for details on these aspects. We also assume a background of linear statistical models and of the general area of design of experiments at an advanced undergraduate level. Excellent accounts of the general area of design of experiments and its applications are available e.g., in Cox (1958), Hinkelmann and Kempthorne (1994), Dean and Voss (1999), Wu and Hamada (2000) and Bailey (2008).

1.2 Outline of the Book

The book has five more chapters followed by an appendix. In Chapter 2, the discussion is initiated by describing the intra-block analysis of an arbitrary block design. Balancing in incomplete block designs is considered next in Section 2.3 of this chapter. The two notions of balance, viz., variance- and efficiency-balance are reviewed. The analysis of incomplete block designs with recovery of inter-block information is discussed in Section 2.4. Finally, in Section 2.5, the notion of efficiency factor of an incomplete block design is briefly studied.

Balanced designs are considered in Chapter 3. The most important of the balanced designs are the classical balanced incomplete block (BIB) designs. Such designs are still found useful in designing experiments in diverse fields and newer applications of these designs, e.g., in visual cryptography, have been found in recent years (see e.g., Bose and Mukerjee (2006), Adhikary, Bose, Kumar and Roy (2007) and the references cited therein). We initiate the discussion in this chapter by considering some properties of BIB designs in Section 3.2. The analysis of BIB designs is briefly considered in Section 3.3. Some results on construction and existence of BIB designs are presented in Section 3.4. Generalizations of BIB designs are considered in the next section. The BIB designs are the only designs in the class of binary, equireplicate and proper designs that are both variance- and efficiency-balanced; however, it is possible to find other variance- and efficiency-balanced designs if one expands the class of designs to non-binary, non-equireplicate or non-proper designs. The construction methods of variance- and efficiency-balanced designs with possibly unequal replications and unequal block sizes are briefly reviewed in Section 3.6. Properties and construction of nested BIB designs are discussed briefly in Section 3.7.

Partially balanced designs are the subject matter of Chapter 4. Among the partially balanced designs, the partially balanced incomplete block (PBIB) designs are the most studied ones and continue to be used in actual applications. These are therefore covered at some length in Sections 4.2–4.6. PBIB designs are formally introduced in Section 4.2 via the notion of an association scheme. The algebra of association matrices is briefly discussed in Section 4.3. Designs with two or more associate classes as also the analysis of PBIB designs are discussed in Sections 4.4–4.6. In Sections 4.7–4.11, some other partially balanced designs which are not necessarily PBIB designs are covered. These include lattice, cyclic, linked block, C designs and α designs.

In Chapters 3 and 4, incomplete block designs are studied for situations where all the treatments are on equal footing and thus, the interest is mainly on elementary treatment contrasts or, more generally, on a complete set of orthonormal treatment contrasts. However, in practice there are situations where the interest lies in inference on contrasts of special types. Such situations arise typically, e.g., in factorial experiments and biological assays. Incomplete block designs for such experiments are considered in Chapter 5. Specifically, incomplete block designs for factorial experiments (Section 5.2), biological assays (Section 5.3), test-control experiments (Section 5.4) and diallel cross experiments (Section 5.5) are covered. Finally, in Section 5.6, results on incomplete block designs that are robust against an outlier and against missing data are reviewed. Some aspects of trend-free block designs are also covered in this section.

In Chapter 6, optimality aspects of some incomplete block designs are discussed. Different optimality criteria are introduced in Section 6.2. Important results on optimality of proper incomplete block designs for inference on a complete set of orthonormal treatment contrasts are reviewed in Section 6.3. Optimal designs for making inferences on contrasts among several test treatments and a control are discussed in Section 6.4. Optimality of designs for parallel line assays, considered in Chapter 5 are reviewed in Section 6.5. In the last section (Section 6.6), optimal incomplete block designs for diallel crosses are considered.

The Appendix consists of four sections. Some results in linear algebra that are used throughout the book are given in Section A.1. In Section A.2, some basic results in linear statistical models are summarized. Section A.3 describes some essential facts about finite (Galois) fields. In Section A.4, basic ideas and results from finite projective and Euclidean geometries are reviewed.

Chapter 2

Analysis and Properties of Block Designs

2.1 Introduction

This chapter is concerned with the analysis and some general properties of arbitrary block designs, including incomplete block designs. We initiate the discussion in Section 2.2 by reviewing the intra-block analysis of a general block design under a standard fixed effects model. Two notions of balance are introduced and studied in Section 2.3. The recovery of inter-block information is discussed in Section 2.4. In section 2.5, the notion of efficiency factor of an incomplete block design is briefly introduced. Throughout, we use the following notations and terminology in respect of an arbitrary block design.

Consider an arbitrary block design d involving v treatments and b blocks. For $1 \leq j \leq b$, the size of the jth block of d is denoted by k_{dj}, that is, the jth block has k_{dj} experimental units and for $1 \leq i \leq v$, r_{di} is the replication of the ith treatment in d, that is, the ith treatment appears r_{di} times in the d. A design d is called *proper* if $k_{dj} = k$ for all j and *equireplicate* if $r_{di} = r$ for all i. A block design d is completely characterized by a $v \times b$ matrix $N_d = (n_{dij})$ where n_{dij} is the number of times the ith treatment appears in the jth block. Clearly, the $\{n_{dij}\}$ are nonnegative integers and

$$\sum_{i=1}^{v} n_{dij} = k_{dj},\ 1 \leq j \leq b,\ \sum_{j=1}^{b} n_{dij} = r_{di},\ 1 \leq i \leq v. \qquad (2.1.1)$$

The matrix N_d is called the *incidence matrix* of the design d. A block design is called *binary* if $n_{dij} = 0$ or 1 for all i, j, $1 \leq i \leq v, 1 \leq j \leq b$.

We shall throughout use the following notations with respect to a block design d:

$$\begin{aligned} \boldsymbol{k}_d &= (k_{d1}, \ldots, k_{db})', \text{ the column vector of block sizes,} \\ \boldsymbol{r}_d &= (r_{d1}, \ldots, r_{dv})', \text{ the column vector of replication numbers,} \\ K_d &= \operatorname{diag}(k_{d1}, \ldots, k_{db}), \text{ the diagonal matrix of block sizes,} \\ R_d &= \operatorname{diag}(r_{d1}, \ldots, r_{dv}), \text{ the diagonal matrix of replication numbers.} \end{aligned}$$

Then, it is easy to see that $\mathbf{1}_v' N_d = \boldsymbol{k}_d'$ and $N_d \mathbf{1}_b = \boldsymbol{r}_d$, where N_d is the incidence matrix of d. The subscript d refers to a given block design d and we may drop this subscript when there is no confusion likely.

2.2 Intra-block Analysis

Consider a block design with v treatments, b blocks and incidence matrix $N_d = (n_{dij})$. As before, for $1 \leq i \leq v$, we let r_{di} to denote the replication of the ith treatment and for $1 \leq j \leq b$, k_{dj}, to denote the block size of the jth block. At this stage, we do not make any assumptions about the block sizes or replications of the treatments. For the analysis of the data obtained through the design d, we postulate the following linear model:

$$Y_{iju} = \mu + \tau_i + \beta_j + \epsilon_{iju}, \tag{2.2.1}$$

where Y_{iju} is the observable random variable corresponding to the uth observation in the (i, j)th cell defined by the ith treatment and the jth block, μ is a general mean, τ_i, the effect of the ith treatment, β_j, the effect of the jth block and $\{\epsilon_{iju}\}$ are random error components, assumed to be mutually uncorrelated with zero means and constant finite variance σ^2. Clearly, if $n_{dij} = 0$ for some pair (i, j), then there is no observation in that cell. Barring the error components, all other effects on the right side of (2.2.1) are assumed to be *fixed* (nonrandom). The analysis under such a fixed effects model is generally termed as *intra-block* analysis.

It may be noted that in the above formulation, we have considered an arbitrary block design which is possibly unequally replicated, has possibly unequal block sizes and could be non-binary. Also, in model (2.2.1), we assume that the intra-block variance, σ^2, is a constant. In practice, often an experimenter chooses a block design with equal block sizes. However, there exist practical situations where blocks of unequal sizes arise quite naturally (see e.g., Pearce (1964)) and, in such situations, one might have to use designs with unequal block sizes. If the intra-block variance is assumed to be proportional to the block size and the block sizes do not vary appreciably, the assumption of constancy of intra-block variance is not very serious and consequently, the analysis that follows under the model (2.2.1) remains valid.

We may rewrite (2.2.1) in matrix notation as

$$\boldsymbol{Y} = \mu \mathbf{1}_n + D'_{1d}\boldsymbol{\tau} + D'_{2d}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \tag{2.2.2}$$

with

$$\mathbb{E}(\boldsymbol{\epsilon}) = \mathbf{0}, \quad \mathbb{D}(\boldsymbol{\epsilon}) = \sigma^2 I_n, \tag{2.2.3}$$

where $\boldsymbol{Y}$ is the $n \times 1$ vector of Y_{iju}'s, $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_v)', \boldsymbol{\beta} = (\beta_1, \ldots, \beta_b)'$, $\boldsymbol{\epsilon}$ is the vector of random error components, $n = \sum_{i=1}^{v} r_{di} = \sum_{j=1}^{b} k_{dj}$; D_{1d} (respectively, D_{2d}) denotes the $v \times n$ (respectively, $b \times n$) treatments (respectively, blocks) *versus* observations incidence matrix, i.e., the (α, β)th element of D_{1d} (respectively, D_{2d}) is 1 if the βth observation comes from the αth treatment (respectively, αth block), and is zero otherwise. In (2.2.3), $\mathbb{E}$ stands for expectation and $\mathbb{D}$ denotes the dispersion (variance-covariance) matrix.

It can easily be verified that

$$D_{1d}D'_{1d} = R_d, D_{2d}D'_{2d} = K_d, D_{1d}D'_{2d} = N_d, \tag{2.2.4}$$

$$D_{1d}\mathbf{1}_n = \boldsymbol{r}_d, D_{2d}\mathbf{1}_n = \boldsymbol{k}_d, D'_{1d}\mathbf{1}_v = \mathbf{1}_n = D'_{2d}\mathbf{1}_b. \tag{2.2.5}$$

Applying the method of least squares for the estimation of parameters of the model (2.2.2), we arrive at the following normal equations:

$$\begin{pmatrix} n & \boldsymbol{k}'_d & \boldsymbol{r}'_d \\ \boldsymbol{k}_d & K_d & N'_d \\ \boldsymbol{r}_d & N_d & R_d \end{pmatrix} \begin{pmatrix} \mu \\ \boldsymbol{\beta} \\ \boldsymbol{\tau} \end{pmatrix} = \begin{pmatrix} G \\ \boldsymbol{B} \\ \boldsymbol{T} \end{pmatrix}, \tag{2.2.6}$$

where $G = \mathbf{1}'_n\boldsymbol{Y}$ is the grand total of all observations, $\boldsymbol{B} = (B_1, \ldots, B_b)' = D_{2d}\boldsymbol{Y}$ is the vector of block totals and $\boldsymbol{T} = (T_1, \ldots, T_v)' = D_{1d}\boldsymbol{Y}$ is the vector of treatment totals.

Observe that (2.2.5) implies

$$Rank \begin{pmatrix} n & \boldsymbol{k}'_d & \boldsymbol{r}'_d \\ \boldsymbol{k}_d & K_d & N'_d \\ \boldsymbol{r}_d & N_d & R_d \end{pmatrix} = Rank(E), \tag{2.2.7}$$

where

$$E = \begin{pmatrix} K_d & N'_d \\ N_d & R_d \end{pmatrix}. \tag{2.2.8}$$

We now have the following result.

Lemma 2.2.1 *For any block design d with v treatments and b blocks, the identity*

$$b + Rank(C_d) = v + Rank(D_d) \tag{2.2.9}$$

holds, where

$$C_d = R_d - N_d K_d^{-1} N_d' \text{ and } D_d = K_d - N_d' R_d^{-1} N_d. \tag{2.2.10}$$

Proof. Let

$$A_1 = \begin{pmatrix} I_b & \mathbf{0} \\ -N_d K_d^{-1} & I_v \end{pmatrix}, A_2 = \begin{pmatrix} I_b & -N_d' R_d^{-1} \\ \mathbf{0} & I_v \end{pmatrix}. \tag{2.2.11}$$

Clearly, A_1 and A_2 are nonsingular matrices. Hence,

$$\begin{aligned} Rank(E) &= Rank(A_1 E A_1') = Rank \begin{pmatrix} K_d & \mathbf{0} \\ \mathbf{0} & C_d \end{pmatrix} \\ &= Rank(A_2 E A_2') = Rank \begin{pmatrix} D_d & \mathbf{0} \\ \mathbf{0} & R_d \end{pmatrix}. \end{aligned} \tag{2.2.12}$$

The result now follows. □

Note that the matrices C_d and D_d, given by (2.2.10) are symmetric of orders v and b, respectively, and each has zero row sums.

Premultiplying both sides of (2.2.6) by the matrix

$$(\mathbf{0}_v \quad -N_d K_d^{-1} \quad I_v)$$

we get the equation

$$C_d \boldsymbol{\tau} = \boldsymbol{Q} \tag{2.2.13}$$

where

$$\boldsymbol{Q} = \boldsymbol{T} - N_d K_d^{-1} \boldsymbol{B}. \tag{2.2.14}$$

The vector $\boldsymbol{Q}$ is called the vector of *adjusted treatment totals.* Equations (2.2.13) are often called the *reduced normal equations for treatment effects.*

The matrix C_d, given by (2.2.10) and appearing in the reduced normal equations (2.2.13), is of fundamental importance in the analysis of block designs. This matrix is generally referred to as the "C-matrix" of the design and we shall often use this terminology in the sequel.

Remark 2.2.1 The matrix C_d can also be expressed in terms of an orthogonal projection matrix (see A.1.12 of the Appendix) as

$$C_d = D_{1d} \mathrm{pr}^{\perp}(L) D_{1d}',$$

where $L = (\mathbf{1}_n \quad D_{2d}')$.

Lemma 2.2.2 $\mathbb{E}(\boldsymbol{Q}) = C_d\boldsymbol{\tau}$; $\mathbb{D}(\boldsymbol{Q}) = \sigma^2 C_d$.

Proof. From (2.2.14),

$$\begin{aligned}
\mathbb{E}(\boldsymbol{Q}) &= \mathbb{E}(\boldsymbol{T} - N_d K_d^{-1}\boldsymbol{B}) \\
&= (D_{1d} - D_{1d}D_{2d}'(D_{2d}D_{2d}')^{-1}D_{2d})\mathbb{E}(\boldsymbol{Y}) \\
&= (D_{1d} - D_{1d}D_{2d}'(D_{2d}D_{2d}')^{-1}D_{2d})(\mu\mathbf{1}_n + D_{1d}'\boldsymbol{\tau} + D_{2d}'\boldsymbol{\beta}) \\
&= (R_d - N_d K_d^{-1}N_d')\boldsymbol{\tau}, \quad \text{using (2.2.4)} \\
&= C_d\boldsymbol{\tau}.
\end{aligned}$$

Again,

$$\begin{aligned}
\sigma^{-2}\mathbb{D}(\boldsymbol{Q}) &= (D_{1d} - D_{1d}D_{2d}'(D_{2d}D_{2d}')^{-1}D_{2d}) \times \\
&\quad\ (D_{1d} - D_{1d}D_{2d}'(D_{2d}D_{2d}')^{-1}D_{2d})' \\
&= R_d - N_d K_d^{-1}N_d' \\
&= C_d.
\end{aligned}$$

In the right side of the first line above, $\times$ denotes the usual matrix product. □

Lemma 2.2.3 *(a) C_d is n.n.d.; (b) $C_d\mathbf{1}_v = \mathbf{0}$ and hence Rank$(C_d) \le v-1$; (c) the equations (2.2.13) are consistent, whatever be the rank of C_d.*

Proof. The proofs of (a) and (b) are left as an exercise; we provide a proof of (c) only. We have

$$\begin{aligned}
C_d &= D_{1d}D_{1d}' - D_{1d}D_{2d}'(D_{2d}D_{2d}')^{-1}D_{2d}D_{1d}' \\
&= D_{1d}(I - D_{2d}'(D_{2d}D_{2d}')^{-1}D_{2d})D_{1d}' \qquad (2.2.15) \\
&= D_{1d}Z_dZ_d'D_{1d}',
\end{aligned}$$

where

$$Z_d = I - D_{2d}'(D_{2d}D_{2d}')^{-1}D_{2d} = \text{pr}^{\perp}(D_{2d}'), \qquad (2.2.16)$$

by virtue of the result in A.1.12, is a symmetric idempotent matrix. From (2.2.15), it follows that

$$\mathcal{C}(C_d) = \mathcal{C}(D_{1d}Z_d) \qquad (2.2.17)$$

where $\mathcal{C}(A)$ denotes the column space of a matrix A and in proving (2.2.17), we have used the fact stated in A.1.4. Also from (2.2.14),

$$\begin{aligned}
\boldsymbol{Q} &= (D_{1d} - D_{1d}D_{2d}'(D_{2d}D_{2d}')^{-1}D_{2d})\boldsymbol{Y} \\
&= D_{1d}(I - D_{2d}'(D_{2d}D_{2d}')^{-1}D_{2d})\boldsymbol{Y} \qquad (2.2.18) \\
&= D_{1d}Z_d\boldsymbol{Y} = D_{1d}\text{pr}^{\perp}(D_{2d}')\boldsymbol{Y}.
\end{aligned}$$

Hence, $Q \in \mathcal{C}(D_{1d}Z_d)$ and the proof is complete. □

Definition 2.2.1 *A linear parametric function $p'\tau$ is said to be a treatment contrast if p is non-null and $p'1_v = 0$. A treatment contrast is called an elementary contrast if p has only two nonzero entries, these being -1 and 1. Furthermore, a treatment contrast $p'\tau$ is called normalized if $p'p = 1$.*

Lemma 2.2.4 *A linear parametric function $p'\tau$ is estimable under a block design d if and only if $p \in \mathcal{C}(C_d)$, where C_d is the C-matrix of d.*

Proof. Suppose $p \in \mathcal{C}(C_d) \Rightarrow p = C_d\lambda$ for some vector λ. If $\hat{\tau}$ is a solution of (2.2.13), then

$$p'\hat{\tau} = \lambda' C_d \hat{\tau} = \lambda' Q$$

and by Lemma 2.2.2,

$$\mathbb{E}(p'\hat{\tau}) = \lambda'\mathbb{E}(Q) = \lambda' C_d \tau = p'\tau,$$

so that $p'\tau$ is estimable.

Conversely, let $p'\tau$ be estimable. Then, by the definition of an estimable function, there exists a linear function of Y, say, $l'Y$ such that

$$\begin{aligned} \mathbb{E}(l'Y) &= p'\tau, \text{ for all } \mu \in \mathbb{R}, \tau \in \mathbb{R}^v, \beta \in \mathbb{R}^b \\ \Rightarrow \mu l'1_n + l'D_{1d}'\tau + l'D_{2d}'\beta &= p'\tau, \text{ for all } \mu \in \mathbb{R}, \tau \in \mathbb{R}^v, \beta \in \mathbb{R}^b \end{aligned}$$

$$\Rightarrow l'1_n = 0, \quad l'D_{2d}' = 0, \quad p' = l'D_{1d}'.$$

Now, $p = D_{1d}l = D_{1d}(I - D_{2d}'(D_{2d}D_{2d}')^{-1}D_{2d})l$, as $D_{2d}l = 0$. This implies that $p \in \mathcal{C}(D_{1d}Z)$. But $\mathcal{C}(D_{1d}Z) = \mathcal{C}(C_d)$ and hence the result. □

It follows from Lemma 2.2.3 (b) and Lemma 2.2.4 that a necessary condition for a linear parametric function $p'\tau$ to be estimable under a design d is that it be a treatment contrast. Also, if $p'\tau$ is estimable under a design d, then its best linear unbiased estimator (BLUE) is $p'\hat{\tau}$, where $\hat{\tau}$ is a solution of (2.2.13).

Definition 2.2.2 *A block design d is said to connected if all treatment contrasts are estimable under d.*

An equivalent definition of connectedness of a block design, given by Bose (1950) is as follows.

Definition 2.2.3 *A treatment i and a block j in a block design are said to be associated if the treatment i appears in block j. A pair of treatments is said to be connected if it is possible to pass from one to the other through a chain consisting alternatively of treatments and blocks such that any two members of a chain are associated. A design is said to be connected if every pair of treatments is connected.*

The property of connectedness is related to the rank of the C-matrix of the design, as shown in the following result.

Theorem 2.2.1 *A block design d with v treatments is connected if and only if $Rank(C_d) = v - 1$.*

Proof. Let d be connected. Then all treatment contrasts are estimable under d, i.e., by Lemma 2.2.4, $\boldsymbol{p} \in \mathcal{C}(C_d)$ for all non-null $v \times 1$ vectors $\boldsymbol{p}$ satisfying $\boldsymbol{p}'\mathbf{1}_v = 0$. Since the space of all such vectors has dimension $v-1$, we get $Rank(C_d) \geq v-1$, which, in conjunction with Lemma 2.2.3 (b), yields $Rank(C_d) = v - 1$.

Conversely, if $Rank(C_d) = v-1$, then by Lemma 2.2.3 (b), $\boldsymbol{p} \in \mathcal{C}(C_d)$ for all non-null $v \times 1$ vectors satisfying $\boldsymbol{p}'\mathbf{1}_v = 0$. Thus by Lemma 2.2.4, all treatment contrasts are estimable and the design is connected. □

We next bring in the notion of an *orthogonal* block design. Consider the equations (2.2.6). Premultiplying both sides of (2.2.6) by the matrix

$$(\mathbf{0}_b \;\; I_b \;\; -N_d'R_d^{-1}),$$

we get the equation

$$D_d\boldsymbol{\beta} = \boldsymbol{P} \tag{2.2.19}$$

where

$$D_d = K_d - N_d'R_d^{-1}N_d, \quad \boldsymbol{P} = \boldsymbol{B} - N_d'R_d^{-1}\boldsymbol{T}. \tag{2.2.20}$$

The equations (2.2.19) are the reduced normal equations for *block effects.* The vector $\boldsymbol{P}$ is called the vector of adjusted block totals. Rewriting

$$\begin{aligned} \boldsymbol{Q} &= \boldsymbol{T} - N_dK_d^{-1}\boldsymbol{B} \\ &= (D_{1d} - D_{1d}D_{2d}'K_d^{-1}D_{2d})\boldsymbol{Y}, \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{P} &= \boldsymbol{B} - N_d'R_d^{-1}\boldsymbol{T} \\ &= (D_{2d} - D_{2d}D_{1d}'R_d^{-1}D_{1d})\boldsymbol{Y} \end{aligned}$$

it is seen that the covariance between $\boldsymbol{Q}$ and $\boldsymbol{P}$ (i.e., the $v \times b$ matrix of covariances between the components of $\boldsymbol{Q}$ and $\boldsymbol{P}$) is

$$\begin{aligned} \sigma^{-2}\text{Cov}(\boldsymbol{Q}, \boldsymbol{P}) &= (D_{1d} - D_{1d}D'_{2d}K_d^{-1}D_{2d})(D_{2d} - D_{2d}D'_{1d}R_d^{-1}D_{1d})' \\ &= N_dK_d^{-1}N'_dR_d^{-1}N_d - N_d. \end{aligned} \tag{2.2.21}$$

Lemma 2.2.5 *For a connected block design d, the covariance between* $\boldsymbol{Q}$ *and* $\boldsymbol{P}$ *is zero if and only if* $N_d = \boldsymbol{r}_d\boldsymbol{k}'_d/n$.

Proof. The "if" part is easy to prove, so we only provide a proof of the "only if" part.

$$\begin{aligned} \text{Cov}(\boldsymbol{Q}, \boldsymbol{P}) &= \mathbf{0} \\ \Rightarrow N_dK_d^{-1}N'_dR_d^{-1}N_d - N_d &= \mathbf{0} \\ \Rightarrow (R_d - C_d)R_d^{-1}N_d - N_d &= \mathbf{0} \\ \Rightarrow C_dR_d^{-1}N_d &= \mathbf{0}. \end{aligned}$$

Let $R_d^{-1}N_d = A$, where A is a $v \times b$ matrix. Since the design is connected, by Lemma 2.2.3 (b) and Theorem 2.2.1, it follows that the columns of A, say $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_b$ are proportional to $\mathbf{1}_v$, i.e.,

$$\boldsymbol{a}_i = \alpha_i \mathbf{1}_v, \quad 1 \le i \le b, \tag{2.2.22}$$

where α_i's are some scalars. This gives

$$A = R_d^{-1}N_d = \mathbf{1}_v\boldsymbol{\alpha}', \tag{2.2.23}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_b)'$. From (2.2.23), we have

$$N_d = R_d\mathbf{1}_v\boldsymbol{\alpha}' = \boldsymbol{r}_d\boldsymbol{\alpha}'$$

which gives

$$\mathbf{1}'_vN_d = \boldsymbol{k}'_d = \mathbf{1}'_v\boldsymbol{r}_d\boldsymbol{\alpha}' = n\boldsymbol{\alpha}'.$$

Hence, we have $\boldsymbol{\alpha}' = \boldsymbol{k}'_d/n$ and the result is proved. □

We now have the following definition.

Definition 2.2.4 *A connected block design is said to be orthogonal if the incidence matrix of the design satisfies the condition of Lemma 2.2.5.*

From (2.2.13), (2.2.19) and Lemma 2.2.5, it follows that under an orthogonal design, the BLUE of any treatment contrast is uncorrelated with the BLUE of any block contrast and, this is a characteristic property of an orthogonal block design. See also Remark 2.2.2.

Designs which are not orthogonal as per Definition 2.2.4 may be termed *nonorthogonal.* It is clear from Lemma 2.2.5 that if *at least* one entry of the incidence matrix N_d of a design d is zero, d cannot be orthogonal. A block design with at least one zero entry in its incidence matrix is called an *incomplete block design.* It is also clear that a randomized *complete* block design is an orthogonal design as, for such designs, the incidence matrix has all its entries equal to 1.

We now discuss the testing problem in the context of *connected* block designs. To that end, we make an additional assumption that observations are normally distributed. Suppose we are interested in testing a hypothesis

$$H : \tau_1 = \tau_2 = \cdots = \tau_v \tag{2.2.24}$$

against the alternative

$$A : \text{At least one pair of treatment effects is different from each other.} \tag{2.2.25}$$

The hypothesis H is equivalent to testing the hypothesis that a set of $(v-1)$ linearly independent treatment contrasts are each equal to zero. Following the standard results in linear models (see Section A.2 of the Appendix), the residual sum of squares under the model (2.2.2) is

$$R_0{}^2 = (\boldsymbol{Y} - \hat{\mu}\mathbf{1}_n - D'_{1d}\hat{\boldsymbol{\tau}} - D'_{2d}\hat{\boldsymbol{\beta}})'(\boldsymbol{Y} - \hat{\mu}\mathbf{1}_n - D'_{1d}\hat{\boldsymbol{\tau}} - D'_{2d}\hat{\boldsymbol{\beta}}),$$

where $\hat{\mu}, \hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\beta}}$ are a solution of the equations (2.2.6). Now,

$$\begin{aligned} R_0{}^2 &= (\boldsymbol{Y} - \hat{\mu}\mathbf{1}_n - D'_{1d}\hat{\boldsymbol{\tau}} - D'_{2d}\hat{\boldsymbol{\beta}})'(\boldsymbol{Y} - \hat{\mu}\mathbf{1}_n - D'_{1d}\hat{\boldsymbol{\tau}} - D'_{2d}\hat{\boldsymbol{\beta}}) \\ &= \boldsymbol{Y}'(\boldsymbol{Y} - \hat{\mu}\mathbf{1}_n - D'_{1d}\hat{\boldsymbol{\tau}} - D'_{2d}\hat{\boldsymbol{\beta}}), \text{ by virtue of normal equations} \\ &= \boldsymbol{Y}'\boldsymbol{Y} - (\boldsymbol{T}' - \boldsymbol{B}'K_d^{-1}N'_d)\hat{\boldsymbol{\tau}} - (\boldsymbol{B}'K_d^{-1}N'_d\hat{\boldsymbol{\tau}} + \boldsymbol{B}'\hat{\boldsymbol{\beta}} + \hat{\mu}G) \\ &= \boldsymbol{Y}'\boldsymbol{Y} - \boldsymbol{Q}'\hat{\boldsymbol{\tau}} - \boldsymbol{B}'K_d^{-1}\boldsymbol{B}, \text{ again, using the normal equations.} \end{aligned} \tag{2.2.26}$$

The residual sum of squares, under the hypothesis H is similarly seen to be

$$R_1{}^2 = \boldsymbol{Y}'\boldsymbol{Y} - \boldsymbol{B}'K_d^{-1}\boldsymbol{B}. \tag{2.2.27}$$

Hence, ${R_1}^2 - {R_0}^2 = \boldsymbol{Q}'\hat{\boldsymbol{\tau}} = \hat{\boldsymbol{\tau}}'\boldsymbol{Q}$. Following the results given in Section A.2 of the Appendix, a test for the hypothesis H is then based on the ratio

$$\mathcal{F} = \frac{({R_1}^2 - {R_0}^2)/(v-1)}{({R_0}^2/n_e)} \tag{2.2.28}$$

where $n_e = n - v - b + 1$, the degrees of freedom associated with ${R_0}^2$. The statistic $\mathcal{F}$, under the hypothesis H, has a central F-distribution on $(v-1)$ and n_e degrees of freedom. One would therefore reject H at α level of significance if $\mathcal{F} > F_{\alpha;v-1,n_e}$, where $F_{\alpha;v-1,n_e}$ is the upper α percent point of an F-distribution on $(v-1)$ and n_e degrees of freedom.

The numerator of the statistic $\mathcal{F}$, viz., $\hat{\boldsymbol{\tau}}'\boldsymbol{Q}$, is sometimes called the "adjusted treatment sum of squares". The above analysis can be put formally in an analysis of variance table, as shown below.

TABLE 2.2.1: Analysis of Variance

Source	d.f.	S.S.
Treatments(adj.)	$v-1$	$\hat{\boldsymbol{\tau}}'\boldsymbol{Q} = S_t^2$
Blocks(unadj.)	$b-1$	$\boldsymbol{B}'K_d^{-1}\boldsymbol{B} - n^{-1}\boldsymbol{Y}'J_n\boldsymbol{Y} = S_{ub}^2$
Intra − block Error	n_e	${R_0}^2$
Total	$n-1$	$\boldsymbol{Y}'(I_n - n^{-1}J_n)\boldsymbol{Y} = S^2$

The sum of squares due to the intra-block error $({R_0}^2)$ is generally obtained by subtracting from the total sum of squares, the total of the sums of squares due to treatments (adjusted) and blocks (unadjusted).

Similarly, if one is interested in testing the hypothesis

$$H' : \beta_1 = \beta_2 = \cdots = \beta_b,$$

one has to obtain, what may be called "adjusted block sum of squares", given by $\sum_{j=1}^{b} \hat{\beta}_j P_j = S_b^2$, say, where $\hat{\beta}_j$ is a solution of (2.2.19). One may obtain the adjusted block sum of squares by using the following identity:

$$S_t^2 + S_{ub}^2 = S_{ut}^2 + S_b^2,$$

where S_{ut}^2 (=Treatment S.S. (unadj.)) is given by

$$S_{ut}^2 = \boldsymbol{T}'R_d^{-1}\boldsymbol{T} - n^{-1}\boldsymbol{Y}'J_n\boldsymbol{Y},$$

and the other terms are as in Table 2.2.1.

Observe that a solution of (2.2.13) is

$$\hat{\boldsymbol{\tau}} = C_d^{-}\boldsymbol{Q}.$$

The adjusted treatment sum of squares is thus

$$\hat{\boldsymbol{\tau}}'\boldsymbol{Q} = \boldsymbol{Q}'\hat{\boldsymbol{\tau}} = \boldsymbol{Q}'C_d^{-}\boldsymbol{Q}. \tag{2.2.29}$$

As shown in the proof of Lemma 2.2.3 (c), $\boldsymbol{Q} \in \mathcal{C}(C_d)$ and thus by A.1.6, $\boldsymbol{Q}'C_d^{-}\boldsymbol{Q}$ is invariant to the choice of a g-inverse of C_d.

If $\boldsymbol{p}'\boldsymbol{\tau}$ is an estimable treatment contrast under a design d, its BLUE, as noted earlier, is $\boldsymbol{p}'\hat{\boldsymbol{\tau}} = \boldsymbol{p}'C_d^{-}\boldsymbol{Q}$ and

$$\begin{aligned}
\operatorname{Var}(\boldsymbol{p}'\hat{\boldsymbol{\tau}}) &= \boldsymbol{p}'C_d^{-}\mathbb{D}(\mathbf{Q})(C_d^{-})'\boldsymbol{p} \\
&= \sigma^2\boldsymbol{p}'C_d^{-}C_d(C_d^{-})'\boldsymbol{p} \\
&= \sigma^2\boldsymbol{p}'C_d^{-}C_d(C_d^{-})'C_d\boldsymbol{\lambda} \quad \text{as } \boldsymbol{p} = C_d\boldsymbol{\lambda}, \quad \text{for some } \boldsymbol{\lambda} \\
&= \sigma^2\boldsymbol{p}'C_d^{-}C_d\boldsymbol{\lambda} \\
&= \sigma^2\boldsymbol{p}'C_d^{-}\boldsymbol{p}.
\end{aligned} \tag{2.2.30}$$

Similarly, if $\boldsymbol{p}'\boldsymbol{\tau}$ and $\boldsymbol{q}'\boldsymbol{\tau}$ are two estimable treatment contrasts, then the covariance between the BLUEs of these contrasts is given by

$$\operatorname{Cov}(\boldsymbol{p}'\hat{\boldsymbol{\tau}}, \boldsymbol{q}'\hat{\boldsymbol{\tau}}) = \sigma^2\boldsymbol{p}'C_d^{-}\boldsymbol{q}. \tag{2.2.31}$$

Remark 2.2.2 From Lemma 2.2.5, it is easy to verify that the C-matrix of an orthogonal design O, denoted by C_O, is given by $C_O = R_O - n^{-1}\boldsymbol{r}_O\boldsymbol{r}_O'$ and a choice of a g-inverse of C_O is $C_O^{-} = R_O^{-1}$, which is very easy to compute; here R_O and $\boldsymbol{r}_O$ are respectively, the diagonal matrix and the column vector of replications in the design O. Under an orthogonal design O, from (2.2.14) and (2.2.29), it is easily seen that the vector of adjusted treatment totals and the adjusted treatment sum of squares are given respectively, by $\boldsymbol{Q}_O = \boldsymbol{T} - n^{-1}G\boldsymbol{r}_O$ and $\boldsymbol{Q}_O'C_O^{-}\boldsymbol{Q}_O = \boldsymbol{T}'R_O^{-1}\boldsymbol{T} - n^{-1}G^2$, where as before, $\boldsymbol{T}, G$ and n are respectively, the vector of treatment totals, the grand total of all observations and the total number of experimental units in O. Thus, under an orthogonal design, the adjusted treatment sum of squares is the same as the unadjusted treatment sum of squares and this makes the computations of the different sums of squares in the analysis of variance table particularly simple. Also, under an orthogonal design, the BLUE of an estimable treatment contrast $\boldsymbol{p}'\boldsymbol{\tau}$, is $\boldsymbol{p}'R_O^{-1}\boldsymbol{Q}_O = \boldsymbol{p}'R_O^{-1}\boldsymbol{T}$ and its variance, by (2.2.30) is, simply $\sigma^2\boldsymbol{p}'R_O^{-1}\boldsymbol{p}$.

From the foregoing, it is clear that the intra-block analysis of an arbitrary block design essentially boils down to the computation of a g-inverse of the C-matrix of the design. For a connected block design d, there are various methods available for the computation of a g-inverse of C_d. Some of these are described below; throughout, the design d is assumed to be connected.

(i) Let $U_d = C_d + \boldsymbol{r}_d\boldsymbol{r}'_d/n$. Then, it can be shown that U_d is nonsingular and that U_d^{-1} is a g-inverse of C_d. Hence a solution of (2.2.13) is given by

$$\hat{\boldsymbol{\tau}} = U_d^{-1}\boldsymbol{Q}. \tag{2.2.32}$$

The adjusted treatment sum of squares is thus $\boldsymbol{Q}'U_d^{-1}\boldsymbol{Q}$.

(ii) A square matrix A is called doubly centered if the row sums and column sums of A are all equal to zero. If a doubly centered matrix A of order m has rank $m-1$, then the unique doubly centered g-inverse of A is A^+, the Moore-Penrose inverse. Rao and Mitra (1971, p. 181) show that under the stated conditions on A, the Moore-Penrose inverse A^+ is given by

$$A^+ = (A + m^{-1}J_m)^{-1} - m^{-1}J_m.$$

Since C_d, the C-matrix of a block design d is doubly centered of order v and $Rank(C_d) = v - 1$ because d is connected,

$$C_d^+ = (C_d + v^{-1}J_v)^{-1} - v^{-1}J_v$$

and furthermore, C_d^+ is also doubly centered.

(iii) Let $\theta_1, \theta_2, \ldots, \theta_m$ $(m \leq v-1)$ be the distinct positive eigenvalues of C_d and $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_m$, the corresponding orthonormal eigenvectors. Then, a g-inverse of C_d is given by

$$C_d^- = \sum_{i=1}^{m} \theta_i^{-1}\boldsymbol{\xi}_i\boldsymbol{\xi}_i'.$$

Remark 2.2.3 In this section, the analysis of data from an arbitrary block design has been discussed under a standard Gauss-Markov model, assuming normality of the errors. As an alternative to this infinite population theory model, one may consider a finite population theory model that takes into account the randomization. Analysis under such a formulation is often termed as randomization analysis. The randomization analysis of block designs has been described in detail in Caliński and

Kageyama (2000) (see also, Hinkelmann and Kempthorne (2005)), and thus we do not elaborate on this analysis here. The above references may be consulted for details on randomization analysis.

2.3 Balancing in Block Designs

In this section, the notion of *balance* in the context of block designs is introduced. Two notions of balance, viz., *variance-balance* and *efficiency-balance* are studied. We begin with a definition of variance-balance.

Definition 2.3.1 *A connected block design is said to be variance-balanced if it permits the estimation of every normalized treatment contrast with the same variance.*

An equivalent definition of variance-balance is as follows.

Definition 2.3.2 *A connected block design is said to be variance-balanced if it permits the estimation of every elementary treatment contrast with the same variance.*

The next result due to Rao (1958) provides a characterization of connected variance-balanced designs.

Theorem 2.3.1 *A connected block design d is variance-balanced if and only if all the nonzero eigenvalues of C_d, the C-matrix of d, are equal.*

Proof. Let $0 = \theta_{d0} < \theta_{d1} \leq \theta_{d2} \leq \cdots \leq \theta_{d,v-1}$ be the eigenvalues of C_d and $\boldsymbol{\xi}_0 = v^{-1/2}\mathbf{1}_v, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_{v-1}$ be the corresponding orthonormal eigenvectors. Define the $v \times (v-1)$ matrix P as

$$P = \begin{bmatrix} \boldsymbol{\xi}_1 & \boldsymbol{\xi}_2 & \cdots & \boldsymbol{\xi}_{v-1} \end{bmatrix}. \tag{2.3.1}$$

Then, $P'\boldsymbol{\tau}$ represents a set of $(v-1)$ orthonormal treatment contrasts, each one of which is estimable under d. The BLUE of $P'\boldsymbol{\tau}$ is $P'\hat{\boldsymbol{\tau}}$, where $\hat{\boldsymbol{\tau}}$ is a solution of the normal equations (2.2.13). The dispersion matrix of $P'\hat{\boldsymbol{\tau}}$ is

$$\begin{aligned} \mathbb{D}(P'\hat{\boldsymbol{\tau}}) &= \sigma^2 P' C_d^- P \\ &= \sigma^2 P' \left(\sum_{i=1}^{v-1} \theta_{di}{}^{-1} \boldsymbol{\xi}_i \boldsymbol{\xi}_i' \right) P \\ &= \sigma^2 \Theta_d^{-1}, \end{aligned} \tag{2.3.2}$$

where $\Theta_d = \mathrm{diag}(\theta_{d1}, \cdots, \theta_{d,v-1})$.

Now, first assume that the design is variance-balanced. Then, since from (2.3.2) we have $\mathrm{Var}(\boldsymbol{\xi}'_i\hat{\boldsymbol{\tau}}) = \sigma^2{\theta_{di}}^{-1}$, $1 \le i \le v-1$, it follows that we must have $\theta_{d1} = \cdots = \theta_{d,v-1}$.

Conversely, if $\theta_{d1} = \cdots = \theta_{d,v-1} = \theta$, say, then by (2.3.2),

$$\mathbb{D}(P'\hat{\boldsymbol{\tau}}) = \frac{\sigma^2}{\theta} I_{v-1}.$$

For any normalized treatment contrast $\boldsymbol{\xi}'\boldsymbol{\tau}$, now observe that $\boldsymbol{\xi} = P\boldsymbol{l}$, where the $(v-1)\times 1$ vector $\boldsymbol{l}$ satisfies $\boldsymbol{l}'\boldsymbol{l} = 1$. Hence

$$\begin{aligned}\mathrm{Var}(\boldsymbol{\xi}'\hat{\boldsymbol{\tau}}) &= \mathrm{Var}(\boldsymbol{l}'P'\hat{\boldsymbol{\tau}}) \\ &= \boldsymbol{l}'\left(\frac{\sigma^2}{\theta} I_{v-1}\right)\boldsymbol{l} \\ &= \frac{\sigma^2}{\theta}.\end{aligned}$$

Thus the variance of the BLUE of every normalized treatment contrast is σ^2/θ, i.e., the design is variance-balanced. □

Corollary 2.3.1 *A connected block design is variance-balanced if and only if its C-matrix is completely symmetric (i.e.,* $C = (a-b)I_v + bJ_v$ *for some scalars* a, b*).*

Proof. The proof is immediate from the following result:

A symmetric matrix A of order n is completely symmetric if and only if A has only two distinct eigenvalues, one of these with multiplicity $n-1$*, and* $\mathbf{1}_n$ *is an eigenvector corresponding to the other eigenvalue.*

To prove this, let A be an $n \times n$ symmetric matrix with distinct eigenvalues θ_1, θ_2, where θ_2 is of multiplicity $n-1$ and $\mathbf{1}_n$ is an eigenvector corresponding to θ_1. Then, there exists an orthogonal matrix $U = \begin{bmatrix} n^{-\frac{1}{2}}\mathbf{1}'_n \\ P \end{bmatrix}$, say, such that

$$UAU' = \begin{bmatrix} \theta_1 & \mathbf{0}' \\ \mathbf{0} & \theta_2 I_{n-1} \end{bmatrix}.$$

Hence,

$$U'UAU'U = A = \theta_1 n^{-1} J_n + \theta_2 P'P = n^{-1}(\theta_1 - \theta_2)J_n + \theta_2 I_n,$$

as, $P'P = I_n - n^{-1}J_n$. The converse is easy to prove. □

From Corollary 2.3.1 and remembering that the row (column) sums of the C-matrix are equal to zero, it is easy to see that the C-matrix of a connected variance-balanced block design d is given by

$$C_d = \theta(I_v - v^{-1}J_v), \tag{2.3.3}$$

where $\theta > 0$ is a scalar.

Since the matrix $(I_v - v^{-1}J_v)$ is idempotent, it is easily seen that the following are two possible choices of a g-inverse of the C-matrix of a variance-balanced design d:
(i) $C_d^- = \theta^{-1}I_v$.
(ii) $C_d^- = \theta^{-1}(I_v - v^{-1}J_v)$.
We next have the following result.

Theorem 2.3.2 *The incidence matrix N_d of an equireplicate, proper, binary variance-balanced design d with v treatments satisfies*

$$N_dN_d' = (r - \lambda)I_v + \lambda J_v, \tag{2.3.4}$$

where r is the replication number of the design, λ is a scalar satisfying $\lambda(v-1) = r(k-1)$ and k is the common block size.

Proof. For an equireplicate and proper block design d with v treatments, b blocks and incidence matrix N_d, the C-matrix is given by

$$C_d = rI_v - k^{-1}N_dN_d'. \tag{2.3.5}$$

Also, if the design is binary, we have

$$\mathrm{tr}(C_d) = vr - vr/k = vr - b. \tag{2.3.6}$$

Since d is variance-balanced, its C-matrix is given by (2.3.3) and thus,

$$\mathrm{tr}(C_d) = \theta(v-1). \tag{2.3.7}$$

From (2.3.6) and (2.3.7), we have

$$\theta = (vr - b)/(v-1). \tag{2.3.8}$$

The result now follows from (2.3.3), (2.3.5) and the fact that $bk = vr$. □

A binary, equireplicate and proper block design with incidence matrix satisfying (2.3.4) is known as a *balanced incomplete block* (BIB) design. Such designs are studied in greater detail in the following chapter.

We now have the following result.

Theorem 2.3.3 *For every nonorthogonal connected equireplicate variance-balanced block design involving v treatments and b blocks, the inequality $b \geq v$ holds.*

Proof. For a connected variance-balanced block design d with v treatments,

$$C_d = \theta(I_v - v^{-1}J_v).$$

Thus, the eigenvalues of C_d are zero (with multiplicity one) and θ with multiplicity $v - 1$. Hence, for an equireplicate design with common replication r, the eigenvalues of the matrix $P_d = N_d K_d^{-1} N_d' = rI_v - C_d$ are r and $r - \theta$, with respective multiplicities 1 and $v - 1$. It follows that P_d is singular if and only if $r = \theta$. In such a case, P_d and hence, N_d is of rank unity and the columns of P_d, and hence those of N_d, are spanned by the vector $\mathbf{1}_v$, the eigenvector corresponding to the zero eigenvalue of C_d. It follows then that if $r = \theta$, the rows of N_d are identical. Clearly, designs with such incidence matrices are orthogonal. If we exclude these orthogonal designs, then for every other equireplicate variance-balanced design, P_d is nonsingular and thus,

$$v = Rank(P_d) = Rank(N_d) \leq b.$$

□

Remark 2.3.1 The inequality $b \geq v$ is referred to as *Fisher's inequality* as Fisher (1940) was the first to have obtained this inequality in the context of BIB designs. The same inequality was obtained by Atiqullah (1961) and Raghavarao (1962) for binary, equireplicate variance-balanced designs. The result in Theorem 2.3.3 due to Dey (1975) generalizes these results.

We next consider the notion of *efficiency-balance.* This notion of balance was introduced by Jones (1959) and studied subsequently by several authors including Caliński (1971), Pearce, Caliński and Marshall (1974), Williams (1975), Puri and Nigam (1975), Dey, Singh and Saha (1981), Caliński and Kageyama (2000) and Dey (2008). Consider a connected block design d with v treatments and b blocks and as before, let C_d be the C-matrix of d. Also, let

$$\begin{aligned} \rho_d &= (\sqrt{r_{d1}}, \ldots, \sqrt{r_{dv}})' \\ R_d^{\frac{1}{2}} &= \text{diag}(\sqrt{r_{d1}}, \ldots, \sqrt{r_{dv}}). \end{aligned}$$

Define the $v \times v$ matrix A_d as

$$A_d = R_d^{-\frac{1}{2}} C_d R_d^{-\frac{1}{2}}, \tag{2.3.9}$$

where $R_d^{-\frac{1}{2}}$ is the inverse of $R_d^{\frac{1}{2}}$.
It can be seen that
(i) $Rank(A_d) = Rank(C_d) = v - 1$,
(ii) A_d is nonnegative definite and,
(iii) the eigenvalues of A_d and $R_d^{-1}C_d$ are the same.

Let $0 = \lambda_{d0} < \lambda_{d1} \leq \lambda_{d2} \leq \cdots \leq \lambda_{d,v-1}$ be the eigenvalues of A_d and $\xi_{d0} = n^{-1/2}\rho_d, \xi_{d1}, \ldots, \xi_{d,v-1}$ be the corresponding orthonormal eigenvectors. Then A_d has the spectral representation

$$A_d = \sum_{i=0}^{v-1} \lambda_{di}\xi_{di}\xi'_{di}. \tag{2.3.10}$$

For $1 \leq i \leq v-1$, the λ_{di}'s are called the *canonical efficiency factors* of the design d (James and Wilkinson (1971)). We now provide a statistical interpretation of the canonical efficiency factors of a (connected) block design d. For $1 \leq i \leq v-1$, let

$$p_i = R_d^{\frac{1}{2}}\xi_{di}. \tag{2.3.11}$$

If $\tau = (\tau_1, \ldots, \tau_v)'$ is the vector of treatment effects, then $p_i'\tau$ represents a contrast of treatment effects for each i, $1 \leq i \leq v-1$, because, $p_i'1_v = \xi'_{di}\rho_d = 0$. The variance of BLUE of $p_i'\tau$ under the design d is then given by

$$\text{Var}(p_i'\hat{\tau})_d = \sigma^2 p_i' C_d^- p_i.$$

In view of (2.3.9) and (2.3.10), choosing

$$C_d^- = R_d^{-\frac{1}{2}} \left(\sum_{i=1}^{v-1} \lambda_{di}^{-1}\xi_{di}\xi'_{di} \right) R_d^{-\frac{1}{2}},$$

we have after some simplification,

$$\text{Var}(p_i'\hat{\tau}) = \sigma^2\lambda_{di}^{-1}, \tag{2.3.12}$$

where, as before, σ^2 is the variance of an observation. Let d_1 be a (possibly hypothetical) orthogonal design with the same replication numbers as in d. Then the variance of the BLUE of $p_i'\tau$ under d_1, from Remark 2.2.2 is

$$\operatorname{Var}(p_i'\hat{\tau})_{d_1} = \sigma^2 p_i' R_d^{-1} p_i = \sigma^2. \tag{2.3.13}$$

If we now define the efficiency factor of the contrast $p_i'\tau$ as the ratio of the variance of the BLUE of $p_i'\tau$ under d_1 to that under d, then one has

$$\text{Efficiency Factor}(p_i'\tau) = \lambda_{di}. \tag{2.3.14}$$

Thus, the canonical efficiency factors of a design d are the efficiency factors of the contrasts $p_i'\tau$ relative to an orthogonal design with the same replication numbers as in d. Note that any treatment contrast is a linear combination of the contrasts $\{p_i'\tau,\ 1 \le i \le v-1\}$ and since these are linearly independent, they form a basis of the contrast space. Pearce, Caliński and Marshall (1974) called a treatment contrast $s'\tau$ a *basic* contrast if and only if $C_d R_d^{-1} s = \epsilon s$ for some positive scalar ϵ. It is not hard to see that the contrast $p_i'\tau$ is a basic contrast for each i, $1 \le i \le v-1$. Thus, the canonical efficiency factors are also the efficiency factors of basic contrasts relative to a comparable orthogonal design. The (overall) efficiency factor of the design is defined to be the harmonic mean of the canonical efficiency factors.
We now have the following two results.

Lemma 2.3.1 *For a connected block design d, all the canonical efficiency factors are in the interval* $(0, 1]$.

Lemma 2.3.2 *For a connected block design, each canonical efficiency factor equals unity if and only if the design is orthogonal.*

The proofs of these two results are left as an exercise. We next have the following definition.

Definition 2.3.3 *A connected block design is called efficiency-balanced if all its canonical efficiency factors are equal to ϵ (say), where $\epsilon \in (0, 1]$.*

The following result due to Williams (1975) provides a characterization of efficiency-balanced designs. The sufficiency part of the result was also obtained by Puri and Nigam (1975).

Theorem 2.3.4 *A connected block design d with $v \ge 3$ treatments is efficiency-balanced if and only if*

$$C_d = \epsilon(R_d - r_d r_d'/n), \tag{2.3.15}$$

where $\epsilon \in (0, 1]$ is a scalar.

Proof. Let d be efficiency-balanced. Then, (2.3.10) becomes

$$A_d = \epsilon \sum_{i=1}^{v-1} \xi_{di}\xi'_{di} = \epsilon \left(I_v - \frac{1}{n}\rho_d\rho'_d \right). \qquad (2.3.16)$$

The proof of the "only if" part is completed by recalling that $C_d = R_d^{\frac{1}{2}} A_d R_d^{\frac{1}{2}}$. The "if" part is easy to prove. □

We remark here that Caliński (1971) and others worked with a matrix M_d (or equivalently, with M_{0d}), where

$$\begin{aligned} M_d &= R_d^{-1} N_d K_d^{-1} N'_d = I - R_d^{-1} C_d, \\ M_{0d} &= M_d - \mathbf{1}\mathbf{r}'_d/n, \end{aligned} \qquad (2.3.17)$$

n being the total number of experimental units in d. Since the eigenvalues of A_d and $R_d^{-1}C_d$ are the same, it follows that the eigenvalues of M_d for a connected efficiency balanced design d are 1 with multiplicity unity and $1 - \epsilon \geq 0$ with multiplicity $v - 1$.

The two notions of balance are different in the sense that a (connected) variance-balanced design need not be efficiency-balanced and vice-versa. For example, consider an incomplete block design with $v = 4$ treatments 1,2,3,4, $b = 4$ blocks and block contents as (1,2), (1,3), (1,4) and (2,3,4). This design is efficiency-balanced with $\epsilon = 3/4$ but not variance-balanced. The following result gives a relationship between the two notions of balance.

Theorem 2.3.5 *If a connected block design d has two of the following three properties, viz., (i) efficiency-balance, (ii) variance-balance, (iii) equal replication, then it has the third.*

Proof. (a) (i) and (iii) $\Rightarrow$ (ii).

Let $r_{di} = r$ for all $i, 1 \leq i \leq v$. Then, by (i) and (iii), the C-matrix of the design d is given by

$$C_d = r\epsilon(I_v - (r/n)J_v) = r\epsilon(I_v - v^{-1}J_v), \qquad (2.3.18)$$

and by (2.3.3), d is variance-balanced.

(b) (ii) and (iii) $\Rightarrow$ (i).

Since d is variance-balanced, we have

$$C_d = \theta(I_v - v^{-1}J_v)$$

where $\theta > 0$ is a scalar and this is of the form (2.3.15), since d is equireplicate. This proves (b).
(c) (i) and (ii) $\Rightarrow$ (iii).

Since d is both efficiency- and variance-balanced, we must have

$$C_d = \epsilon(R_d - \boldsymbol{r}_d\boldsymbol{r}'_d/n) = \theta(I_v - v^{-1}J_v).$$

This holds if and only if $r_{di} = r$ for all $i, 1 \le i \le v$. □

The next result characterizes proper, binary efficiency-balanced designs.

Theorem 2.3.6 *In the class of connected, proper, binary block designs, a balanced incomplete block design (if it exists) is the only efficiency-balanced design.*

Proof. Let d be a proper, binary, efficiency-balanced design. Since d is efficiency-balanced, we have

$$\begin{aligned} C_d &= \epsilon(R_d - \boldsymbol{r}_d\boldsymbol{r}'_d/n) \\ \Rightarrow P_d = N_dK_d^{-1}N'_d &= (1-\epsilon)R_d + \epsilon\boldsymbol{r}_d\boldsymbol{r}'_d/n. \end{aligned}$$

Also, since d is proper, $K_d = kI_b$ and thus

$$P_d = N_dN'_d/k = (1-\epsilon)R_d + \epsilon\boldsymbol{r}_d\boldsymbol{r}'_d/n. \tag{2.3.19}$$

Invoking the fact that d is also binary, the above implies that r_{di}'s are all equal. Thus any proper, binary efficiency-balanced design is equireplicate and hence, by Theorem 2.3.5, is variance-balanced as well. By Theorem 2.3.2, the only binary, proper, variance-balanced design is a balanced incomplete block design (provided it exists). □

In the next result, it is shown that the Fisher's inequality also holds for all non-orthogonal efficiency-balanced designs.

Theorem 2.3.7 *The Fisher's inequality $b \ge v$ holds for all connected non-orthogonal efficiency-balanced designs.*

Proof. By Lemma 2.3.2, an efficiency-balanced design has $\epsilon = 1$ if and only if the design is orthogonal. For an orthogonal design d, $M_d = \mathbf{1}\boldsymbol{r}'_d/n$, which is singular. As observed earlier, for a connected, non-orthogonal efficiency balanced design, M_d has two distinct eigenvalues, 1 and $1-\epsilon > 0$ with multiplicities 1 and $(v-1)$ and hence, in such a case, M_d is nonsingular, which implies that

$$v = Rank(M_d) = Rank(R_d^{-1}N_dK_d^{-1}N_d') = Rank(N_d) \leq b.$$

□

For some more results on efficiency-balanced designs, a reference may be made to Caliński and Kageyama (2000).

2.4 Recovery of Inter-block Information

In the intra-block analysis of block designs discussed in Section 2.2, the treatment and block effects are treated as *fixed*. If the block effects are regarded as random variables, the analysis gets slightly modified as, now one has a *mixed effects* model in contrast to the fixed effects model considered in the intra-block analysis. Under the assumption that the block effects are random, the analysis is termed as *inter-block* analysis, or, analysis with *recovery of inter-block information.*

The need for recovery of inter-block information in incomplete block designs was first felt by Yates (1939, 1940). Yates observed that since the allocation of treatments to incomplete blocks is at random, it is reasonable to assume that the block effects are themselves random variables, rather than fixed. Random block effects can also arise naturally in certain practical situations. For example, suppose an experiment consists of making measurements on a number of machines, the measurements being made by a set of operators. Due to practical limitations, each operator can make observations on a subset of the machines. The operators, which form the incomplete blocks, are possibly a random sample from a population of operators and thus, in such a situation, it is meaningful to treat the block effects as random rather than fixed.

It was observed by Yates that if the experimental material is fairly heterogeneous, treating the block effects as fixed effects might result in the loss of information residing in the contrasts among block totals. To elaborate, suppose a proper incomplete block design with $v=6$ treatments is conducted with the (common) block size $k=3$. Let the block contents be (1,2,4), (2,3,5) and (3,4,6) and let Y_{ij} denote the observation pertaining to the ith treatment in the jth block, $1 \leq i \leq 6, 1 \leq j \leq 3$. The block totals are then

$$B_1 \;=\; Y_{11} + Y_{21} + Y_{41}$$

$$\begin{aligned} B_2 &= Y_{22} + Y_{32} + Y_{52} \\ B_3 &= Y_{33} + Y_{43} + Y_{63}. \end{aligned}$$

Under the model (2.2.1), we then have, for example,

$$\begin{aligned} B_1 - B_2 &= (\tau_1 + \tau_2 + \tau_4 - \tau_2 - \tau_3 - \tau_5) + 3\beta_1 - 3\beta_2 \\ &\quad + (\epsilon_{11} + \epsilon_{21} + \epsilon_{41} - \epsilon_{22} - \epsilon_{32} - \epsilon_{52}). \end{aligned}$$

If we now assume that the block effects $\beta_j, j = 1, 2$, are random variables, each with zero expectation, then $\mathbb{E}(B_1 - B_2) = \tau_1 + \tau_4 - \tau_3 - \tau_5$. A similar picture emerges if we consider other contrasts of the block totals. Thus, under the assumption of random block effects with zero expectations, the block totals contain information about the treatment contrasts. Yates (1939, 1940) presented a method of analysis for recovering the inter-block information in certain block designs. A formal theory for recovery of inter-block information was given by Nair (1944) and a general treatment of the subject is due to Rao (1947b).

In this section, we take up the combined intra-inter-block analysis of proper block designs. To that end, we present two basic approaches. The first approach, which is applicable to any proper block design, consists of finding two estimators of a treatment contrast, one based on the intra-block model and the other on the inter-block model and then combining the two estimators so obtained in an optimal fashion. The second approach due to Bose (1975) gives a unified method of obtaining combined intra-inter-block estimators of treatment contrasts under a *binary* and proper block design.

We postulate the model (2.2.2) with assumptions (2.2.3), and the following *additional* assumptions:

$$\mathbb{E}(\boldsymbol{\beta}) = \mathbf{0}, \quad \mathbb{D}(\boldsymbol{\beta}) = \sigma_b{}^2 I_b, \quad \mathrm{Cov}(\boldsymbol{\beta}, \boldsymbol{\epsilon}) = \mathbf{0}. \tag{2.4.1}$$

This means that the block effects in model (2.2.2) are now uncorrelated random variables with means zero and variance $\sigma_b{}^2$ and, the block effects are also uncorrelated with the error terms. We shall call the model (2.2.2) with the assumptions (2.2.3) and (2.4.1) as the inter-block model.

Let d be a proper block design with common block size k and incidence matrix N_d. As pointed out earlier, since under the inter-block model, the block totals have information on treatment contrasts, it is convenient to work with block totals rather than individual observations. Thus, we now consider the vector of block totals, $\boldsymbol{B}$ and model this, using (2.2.2) as

$$B = D_{2d}Y = k\mu\mathbf{1}_b + N_d'\tau + k\beta + D_{2d}\epsilon, \tag{2.4.2}$$

where D_{2d}, as before, is the $b \times n$ incidence matrix of blocks versus observations and all other terms are as in (2.2.2).

We then have, by virtue of the assumptions (2.2.3) and (2.4.1),

$$\begin{aligned} \mathbb{E}(B) &= k\mu\mathbf{1}_b + N_d'\tau, \\ \mathbb{D}(B) &= (k^2\sigma_b^2 + k\sigma^2)I_b \\ &= k\left(1 + k\frac{\sigma_b^2}{\sigma^2}\right)\sigma^2 I_b \\ &= \sigma^2 W, \end{aligned} \tag{2.4.3}$$

where $W = k(1 + k\sigma_b^2/\sigma^2)I_b = k\rho I_b$ and

$$\rho = (1 + k\sigma_b^2/\sigma^2). \tag{2.4.4}$$

Let us now define the $b \times 1$ vector of "observations" as

$$\begin{aligned} Z &= W^{-\frac{1}{2}}B \\ &= \frac{1}{(k\rho)^{\frac{1}{2}}}B. \end{aligned} \tag{2.4.5}$$

It is then easy to verify that

$$\mathbb{D}(Z) = \sigma^2 I. \tag{2.4.6}$$

The inter-block model in terms of the "observations" Z can now be written as

$$\mathbb{E}(Z) = W^{-\frac{1}{2}}(k\mathbf{1}_b \; N_d')\begin{pmatrix}\mu \\ \tau\end{pmatrix}, \quad \mathbb{D}(Z) = \sigma^2 I. \tag{2.4.7}$$

The normal equations under the model (2.4.7) are then given by

$$\begin{pmatrix} k\mathbf{1}_b' \\ N_d \end{pmatrix} W^{-1}(k\mathbf{1}_b \; N_d')\begin{pmatrix}\mu \\ \tau\end{pmatrix} = \begin{pmatrix} k\mathbf{1}_b' \\ N_d \end{pmatrix} W^{-1}B. \tag{2.4.8}$$

This simplifies to

$$\begin{pmatrix} bk^2 & kr_d' \\ kr_d & N_dN_d' \end{pmatrix}\begin{pmatrix}\mu \\ \tau\end{pmatrix} = \begin{pmatrix} k\mathbf{1}_b' \\ N_d \end{pmatrix} B = \begin{pmatrix} kG \\ N_dB \end{pmatrix}, \tag{2.4.9}$$

where, as before, $\boldsymbol{r}_d = (r_{d1}, \ldots, r_{dv})'$ is the vector of replication numbers in d, $\boldsymbol{B} = (B_1, \ldots, B_b)'$ is the vector of block totals and G is the grand total of all observations. Premultiplying both sides of (2.4.9) by the nonsingular matrix

$$\begin{bmatrix} 1 & \mathbf{0}_v' \\ -(bk)^{-1}\boldsymbol{r}_d & I_v \end{bmatrix}$$

we obtain the following equations:

$$\begin{bmatrix} bk^2 & k\boldsymbol{r}_d' \\ \mathbf{0} & N_dN_d' - \boldsymbol{r}_d\boldsymbol{r}_d'/b \end{bmatrix} \begin{bmatrix} \mu \\ \boldsymbol{\tau} \end{bmatrix} = \begin{bmatrix} kG \\ N_d\boldsymbol{B} - G\boldsymbol{r}_d/b \end{bmatrix}. \tag{2.4.10}$$

A solution of (2.4.10) provides inter-block estimates. In particular, if one assumes that the design d is such that N_dN_d' is nonsingular (note that this assumption does *not* hold for all incomplete block designs), then a solution of (2.4.10) can be obtained explicitly by taking the side restriction $\boldsymbol{r}_d'\boldsymbol{\tau} = 0$ as follows:

$$\begin{aligned} \hat{\mu} &= G/bk, \\ \hat{\boldsymbol{\tau}} &= (N_dN_d')^{-1}(N_d\boldsymbol{B} - G\boldsymbol{r}_d/b) \\ &= (N_dN_d')^{-1}(N_d\boldsymbol{B} - (G/bk)N_dN_d'\mathbf{1}_v) \\ &= (N_dN_d')^{-1}N_d\boldsymbol{B} - (G/bk)\mathbf{1}_v, \end{aligned} \tag{2.4.11}$$

the third line above being the consequence of the fact that $N_dN_d'\mathbf{1}_v = k\boldsymbol{r}_d$.

Let $\xi = \boldsymbol{p}'\boldsymbol{\tau}$ be a contrast of treatment effects. Recall from Section 2.2 that the intra-block estimator of ξ under the design d is

$$\hat{\xi_1} = \boldsymbol{p}'C_d^-\boldsymbol{Q}, \tag{2.4.12}$$

with variance

$$\mathrm{Var}(\hat{\xi_1}) = \sigma^2\boldsymbol{p}'C_d^-\boldsymbol{p}. \tag{2.4.13}$$

The inter-block estimator of ξ from (2.4.11) (assuming that d is such that N_dN_d' is nonsingular) is

$$\hat{\xi_2} = \boldsymbol{p}'(N_dN_d')^{-1}N_d\boldsymbol{B}, \tag{2.4.14}$$

with variance

$$\mathrm{Var}(\hat{\xi_2}) = \sigma_1^2\boldsymbol{p}'(N_dN_d')^{-1}\boldsymbol{p} \tag{2.4.15}$$

where $\sigma_1^2 = k(k\sigma_b^2 + \sigma^2)$. It is easily seen that $\hat{\xi_1}$ and $\hat{\xi_2}$ are uncorrelated. If we wish to combine these two uncorrelated estimators of ξ to construct an estimator of ξ with the smallest variance, then the combined estimator is given by the weighted average of $\hat{\xi_1}$ and $\hat{\xi_2}$, the weights being the reciprocal of the respective variances. The combined estimator, say ξ^* is thus given by

$$\xi^* = (\phi_1\hat{\xi_1} + \phi_2\hat{\xi_2})/(\phi_1 + \phi_2), \tag{2.4.16}$$

where $\phi_1 = (\sigma^2 \boldsymbol{p}' C_d^- \boldsymbol{p})^{-1}$ and $\phi_2 = (\sigma_1^2 \boldsymbol{p}'(N_d N_d')^{-1}\boldsymbol{p})^{-1}$. The weights ϕ_1 and ϕ_2 will generally be unknown in practice and have to be estimated from the data. We shall return to this issue later in this section.

We next consider an alternative approach for obtaining combined intra-inter-block estimators of treatment contrasts, essentially following Bose (1975). In this approach, we consider a *binary*, proper block design d with common block size k. Let the observations (obtained via this design) in the vector $\boldsymbol{Y}$ be so arranged that the first k components in $\boldsymbol{Y}$ come from the first block of the design, the next k come from the second block, ..., the last k come from the last block. With this ordering of the observations, it is easy to see that

$$D_{2d}' D_{2d} = I_b \otimes J_k. \tag{2.4.17}$$

Hence, the dispersion matrix of $\boldsymbol{Y}$ is given by

$$\begin{aligned} \mathbb{D}(\boldsymbol{Y}) &= \sigma^2 I_n + \sigma_b{}^2 D_{2d}' D_{2d} \\ &= \sigma^2 (I_b \otimes I_k) + \sigma_b{}^2 (I_b \otimes J_k) \\ &= I_b \otimes L = \Sigma, \text{ say}, \end{aligned} \tag{2.4.18}$$

where

$$L = \sigma^2 I_k + \sigma_b{}^2 J_k. \tag{2.4.19}$$

Clearly,

$$\Sigma^{-1} = I_b \otimes L^{-1}, \tag{2.4.20}$$

where

$$L^{-1} = \alpha I_k + \beta J_k, \tag{2.4.21}$$

$$\alpha = 1/\sigma^2, \beta = -\sigma_b{}^2/\{\sigma^2(\sigma^2 + k\sigma_b{}^2)\}. \tag{2.4.22}$$

Let

$$\omega_1 = 1/\sigma^2, \quad \omega_2 = (\sigma^2 + k\sigma_b{}^2)^{-1}. \tag{2.4.23}$$

We then have

$$\alpha = \omega_1, \quad \beta = -(\omega_1 - \omega_2)/k, \tag{2.4.24}$$

and hence,

$$\begin{aligned}\Sigma^{-1} &= I_b \otimes [\omega_1 I_k - \{(\omega_1 - \omega_2)/k\} J_k] & (2.4.25)\\ &= \omega_1 I_n - \{(\omega_1 - \omega_2)/k\}(I_b \otimes J_k) & (2.4.26)\\ &= \omega_1 I_n - \{(\omega_1 - \omega_2)/k\} D'_{2d} D_{2d}. & (2.4.27)\end{aligned}$$

The inter-block model can be rewritten as

$$\mathbb{E}(\boldsymbol{Y}) = A\boldsymbol{\theta}, \quad \mathbb{D}(\boldsymbol{Y}) = \Sigma, \tag{2.4.28}$$

where

$$A = (\mathbf{1}_n \ \ D'_{1d}), \quad \boldsymbol{\theta}' = (\mu \ \ \boldsymbol{\tau}'). \tag{2.4.29}$$

Following the results in Section A.2 of the Appendix, the least squares normal equations for $\boldsymbol{\theta}$ are

$$A'\Sigma^{-1}A\boldsymbol{\theta} = A'\Sigma^{-1}\boldsymbol{Y},$$

or,

$$\begin{pmatrix} \mathbf{1}'_n\Sigma^{-1}\mathbf{1}_n & \mathbf{1}'_n\Sigma^{-1}D'_{1d} \\ D_{1d}\Sigma^{-1}\mathbf{1}_n & D_{1d}\Sigma^{-1}D'_{1d} \end{pmatrix} \begin{pmatrix} \mu \\ \boldsymbol{\tau} \end{pmatrix} = \begin{pmatrix} \mathbf{1}'_n\Sigma^{-1}\boldsymbol{Y} \\ D_{1d}\Sigma^{-1}\boldsymbol{Y} \end{pmatrix}. \tag{2.4.30}$$

With Σ^{-1} given by (2.4.20) and (2.4.25)–(2.4.27), we have the following simplified expressions for the entries of the coefficient matrix and the right side of the normal equations (2.4.30):

(i) From (2.4.20),

$$\begin{aligned}\mathbf{1}'_n\Sigma^{-1}\mathbf{1}_n &= (\mathbf{1}'_b \otimes \mathbf{1}'_k)(I_b \otimes L^{-1})(\mathbf{1}_b \otimes \mathbf{1}_k)\\ &= (\mathbf{1}'_b \otimes \mathbf{1}'_k)(\mathbf{1}_b \otimes L^{-1}\mathbf{1}_k)\\ &= (\mathbf{1}'_b\mathbf{1}_b) \otimes (\mathbf{1}'_k L^{-1}\mathbf{1}_k) = b(\mathbf{1}'_k L^{-1}\mathbf{1}_k)\\ &= bk(\alpha + k\beta) = n\omega_2. & (2.4.31)\end{aligned}$$

(ii) Using (2.4.27), we have

$$\begin{aligned}\mathbf{1}'_n\Sigma^{-1}D'_{1d} &= \mathbf{1}'_n(\omega_1 I_n - k^{-1}(\omega_1 - \omega_2)D'_{2d}D_{2d})D'_{1d}\\ &= \omega_1 \boldsymbol{r}'_d - (\omega_1 - \omega_2)\boldsymbol{r}'_d\\ &= \omega_2 \boldsymbol{r}'_d. & (2.4.32)\end{aligned}$$

(iii)

$$\begin{aligned}D_{1d}\Sigma^{-1}D'_{1d} &= D_{1d}(\omega_1 I_n - k^{-1}(\omega_1 - \omega_2)D'_{2d}D_{2d})D'_{1d}\\ &= \omega_1 R_d - k^{-1}(\omega_1 - \omega_2)N_d N'_d. & (2.4.33)\end{aligned}$$

(iv)

$$\begin{aligned}\mathbf{1}_n'\Sigma^{-1}\boldsymbol{Y} &= \mathbf{1}_n'(\omega_1 I_n - k^{-1}D_{2d}'D_{2d})\boldsymbol{Y}\\ &= \omega_1 G - k^{-1}(\omega_1-\omega_2)\mathbf{1}_n'D_{2d}'\boldsymbol{B}\\ &= \omega_1 G - k^{-1}(\omega_1-\omega_2)k\mathbf{1}_b'\boldsymbol{B}\\ &= \omega_2 G. \end{aligned}\tag{2.4.34}$$

(v)

$$\begin{aligned}D_{1d}\Sigma^{-1}\boldsymbol{Y} &= D_{1d}(\omega_1 I_n - k^{-1}(\omega_1-\omega_2)D_{2d}'D_{2d})\boldsymbol{Y}\\ &= \omega_1\boldsymbol{T} - k^{-1}(\omega_1-\omega_2)N_d\boldsymbol{B}\\ &= \omega_1\boldsymbol{Q} + \omega_2(\boldsymbol{T}-\boldsymbol{Q}). \end{aligned}\tag{2.4.35}$$

Thus the normal equations (2.4.30) boil down to

$$\begin{pmatrix} n\omega_2 & \omega_2\boldsymbol{r}_d' \\ \omega_2\boldsymbol{r}_d & \omega_1 R_d - (\omega_1-\omega_2)N_dN_d'/k \end{pmatrix}\begin{pmatrix}\mu\\ \boldsymbol{\tau}\end{pmatrix} = \begin{pmatrix}\omega_2 G\\ \boldsymbol{U}\end{pmatrix}, \tag{2.4.36}$$

where $\boldsymbol{U} = \omega_1\boldsymbol{Q} + \omega_2(\boldsymbol{T}-\boldsymbol{Q})$.

Eliminating μ from (2.4.36), we get the reduced normal equations involving $\boldsymbol{\tau}$ only as

$$(\omega_1 C_d + \omega_2 C_d^*)\boldsymbol{\tau} = \omega_1\boldsymbol{Q} + \omega_2\boldsymbol{Q}^*, \tag{2.4.37}$$

where

$$C_d^* = k^{-1}N_dN_d' - n^{-1}\boldsymbol{r}_d\boldsymbol{r}_d', \quad \boldsymbol{Q}^* = \boldsymbol{T} - \boldsymbol{Q} - n^{-1}G\boldsymbol{r}_d \tag{2.4.38}$$

and C_d and $\boldsymbol{Q}$ are as defined in Section 2.2. The normal equations (2.4.37) can also be written as $C_{d_M}\boldsymbol{\tau} = \boldsymbol{Q}_M$, where

$$\begin{aligned} C_{d_M} &= (\omega_1-\omega_2)C_{d_F} + \omega_2 C_O,\\ \boldsymbol{Q}_M &= (\omega_1-\omega_2)\boldsymbol{Q}_F + \omega_2\boldsymbol{Q}_O, \end{aligned}\tag{2.4.39}$$

C_{d_F} (respectively, $\boldsymbol{Q}_F$) is the C-matrix (respectively, the vector of adjusted treatment totals) under a fixed effects model and C_O (respectively, $\boldsymbol{Q}_O$) denote the same under an orthogonal design (recall Remark 2.2.2).

Equations (2.4.37) are the combined inter-intra block reduced normal equations for estimating linear functions of treatment effects. It is easy to verify that

$$C_d^*\mathbf{1}_v = \mathbf{0}, \quad \mathbf{1}_v'\boldsymbol{Q}^* = 0. \tag{2.4.40}$$

From (2.4.37), it is clear that the best linear unbiased estimator of an estimable linear function, say $\boldsymbol{l}'\boldsymbol{\tau}$, of the treatment effects alone is of the form

$$\boldsymbol{q}'(\omega_1\boldsymbol{Q}+\omega_2\boldsymbol{Q}^*),$$

for some $v\times 1$ vector $\boldsymbol{q}$, where $\boldsymbol{q}$ is determined by

$$(\omega_1 C_d+\omega_2 C_d^*)\boldsymbol{q}=\boldsymbol{l}.$$

We now derive the expectations, variances and covariances of Q_i's and Q_i^*'s under the *inter-block* model.

(a)

$$\begin{aligned}\mathbb{E}(\boldsymbol{Q}) &= \mathbb{E}[(D_{1d}-k^{-1}N_dD_{2d})\boldsymbol{Y}]\\ &= (D_{1d}-k^{-1}N_dD_{2d})(\mu\mathbf{1}_n+D_{1d}'\boldsymbol{\tau})\\ &= (D_{1d}-k^{-1}N_dD_{2d})D_{1d}'\boldsymbol{\tau}\\ &= R_d\boldsymbol{\tau}-k^{-1}N_dN_d'\boldsymbol{\tau}=C_d\boldsymbol{\tau}.\end{aligned}\tag{2.4.41}$$

Thus, the expectation of $\boldsymbol{Q}$ is the same, both under the intra-block model and the inter-block model.

(b)

$$\begin{aligned}\mathbb{D}(\boldsymbol{Q}) &= (D_{1d}-k^{-1}N_dD_{2d})(\sigma^2I_n+{\sigma_b}^2D_{2d}'D_{2d})\times\\ &\quad (D_{1d}'-k^{-1}D_{2d}'N_d')\\ &= \sigma^2(R_d-k^{-1}N_dN_d')=\sigma^2C_d=\omega_1^{-1}C_d,\end{aligned}\tag{2.4.42}$$

which is the same as under the intra-block model.

(c)

$$\begin{aligned}\mathbb{E}(\boldsymbol{Q}^*) &= \mathbb{E}(\boldsymbol{T}-\boldsymbol{Q}-n^{-1}G\boldsymbol{r}_d)\\ &= \mathbb{E}(k^{-1}N_d\boldsymbol{B}-n^{-1}G\boldsymbol{r}_d)\\ &= \mathbb{E}((k^{-1}N_dD_{2d}-n^{-1}\boldsymbol{r}_d\mathbf{1}_n')\boldsymbol{Y})\\ &= (k^{-1}N_dD_{2d}-n^{-1}\boldsymbol{r}_d\mathbf{1}_n')(\mu\mathbf{1}_n+D_{1d}'\boldsymbol{\tau})\\ &= (k^{-1}N_dD_{2d}-n^{-1}\boldsymbol{r}_d\mathbf{1}_n')D_{1d}'\boldsymbol{\tau}\\ &= (k^{-1}N_dN_d'-n^{-1}\boldsymbol{r}_d\boldsymbol{r}_d')\boldsymbol{\tau}\\ &= C_d^*\boldsymbol{\tau}.\end{aligned}\tag{2.4.43}$$

(d)

$$\begin{aligned}\mathbb{D}(\boldsymbol{Q}^*) &= (k^{-1}N_dD_{2d}-n^{-1}\boldsymbol{r}_d\mathbf{1}_n')(\sigma^2I_n+{\sigma_b}^2D_{2d}'D_{2d})\times\\ &\quad (k^{-1}D_{2d}'N_d'-n^{-1}\mathbf{1}_n\boldsymbol{r}_d')\\ &= \sigma^2C_d^*+k{\sigma_b}^2C_d^*,\ \text{ on simplification}\\ &= {\omega_2}^{-1}C_d^*.\end{aligned}\tag{2.4.44}$$

On similar lines, one can show that

$$\mathrm{Var}(G) = n\sigma^2 + nk{\sigma_b}^2 = n{\omega_2}^{-1}, \tag{2.4.45}$$
$$\mathrm{Cov}(\boldsymbol{Q}, G) = \boldsymbol{0}, \tag{2.4.46}$$
$$\mathrm{Cov}(\boldsymbol{Q}^*, G) = \boldsymbol{0}, \tag{2.4.47}$$
$$\mathrm{Cov}(\boldsymbol{Q}, \boldsymbol{Q}^*) = \boldsymbol{0}. \tag{2.4.48}$$

Therefore, the dispersion matrix of $(\boldsymbol{Q}, \boldsymbol{Q}^*, G)'$ is

$$\mathbb{D}\begin{pmatrix} \boldsymbol{Q} \\ \boldsymbol{Q}^* \\ G \end{pmatrix} = \begin{pmatrix} \omega_1^{-1}C_d & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & {\omega_2}^{-1}C_d^* & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & n{\omega_2}^{-1} \end{pmatrix}. \tag{2.4.49}$$

It follows then that

$$\mathbb{D}(\omega_1\boldsymbol{Q} + \omega_2\boldsymbol{Q}^*) = \omega_1 C_d + \omega_2 C_d^*. \tag{2.4.50}$$

In practice, the weights ω_1 and ω_2 will rarely be known. These therefore have to be estimated from the data. We take up this aspect now in the context of proper, binary designs. The suggestion of Yates (1940) to estimate the weights ω_1 and ω_2 was to first obtain unbiased estimators of σ^2 and ${\sigma_b}^2$ and then plug in these estimators in the expressions for ω_1 and ω_2 to get the estimators of the weights.

Consider the following sums of squares (S.S) from the intra-block analysis of variance table:

$$\begin{aligned} S_{ut}^2 &= \text{Unadjusted Treatment S.S.} \\ &= \sum_{i=1}^{v} {T_i}^2/r_{di} - G^2/n \\ &= \boldsymbol{T}'R_d^{-1}\boldsymbol{T} - \boldsymbol{Y}'\boldsymbol{1}_n\boldsymbol{1}_n'\boldsymbol{Y}/n \\ &= \boldsymbol{Y}'(D_{1d}'R_d^{-1}D_{1d} - J_n/n)\boldsymbol{Y}. \end{aligned} \tag{2.4.51}$$

$$\begin{aligned} S_{ub}^2 &= \text{Unadjusted Block S.S} \\ &= \boldsymbol{Y}'(D_{2d}'D_{2d}/k - J_n/n)\boldsymbol{Y}. \end{aligned} \tag{2.4.52}$$

$$\begin{aligned} {S_t}^2 &= \text{Adjusted Treatment S.S.} \\ &= \boldsymbol{Q}'C_d^-\boldsymbol{Q}. \end{aligned} \tag{2.4.53}$$

$$S^2 = \text{Total S.S.} = \boldsymbol{Y}'(I_n - J_n/n)\boldsymbol{Y}. \tag{2.4.54}$$

$$\begin{aligned} {S_b}^2 &= \text{Adjusted Block S.S.} \\ &= {S_t}^2 + S_{ub}^2 - S_{ut}^2. \end{aligned} \tag{2.4.55}$$

$$\begin{aligned} {R_0}^2 &= \text{Intra-block Error S.S.} \\ &= S^2 - S_{ut}^2 - {S_b}^2 \\ &= S^2 - {S_t}^2 - S_{ub}^2. \end{aligned} \tag{2.4.56}$$

Using Lemma A.2.1, we evaluate the expectations of the various sums of squares listed earlier.

(i)

$$\begin{aligned} \mathbb{E}(S_{ut}^2) &= \mathbb{E}[\boldsymbol{Y}'(D_{1d}'R_d^{-1}D_{1d} - J_n/n)\boldsymbol{Y}] \\ &= (\mu\mathbf{1}_n' + \boldsymbol{\tau}'D_{1d})(D_{1d}'R_d^{-1}D_{1d} - J_n/n)(\mu\mathbf{1}_n + D_{1d}'\boldsymbol{\tau}) \\ &\quad +\text{tr}[(D_{1d}'R_d^{-1}D_{1d} - J_n/n)(\sigma^2 I_n + \sigma_b^2 D_{2d}'D_{2d})] \\ &= \boldsymbol{\tau}'(R_d - \boldsymbol{r}_d\boldsymbol{r}_d'/n)\boldsymbol{\tau} + (v-1)\sigma^2 - k{\sigma_b}^2 \\ &\quad +{\sigma_b}^2\text{tr}(N_d'R_d^{-1}N_d). \end{aligned} \tag{2.4.57}$$

Since we are considering only *binary* designs,

$$\begin{aligned} \text{tr}(N_d'R_d^{-1}N_d) &= \sum_i\sum_j {n_{dij}}^2/r_{di} \\ &= \sum_i r_{di}^{-1}\sum_j n_{dij}^2 = \sum_i r_{di}^{-1}\sum_j n_{dij} \\ &= \sum_i r_{di}^{-1}r_{di} = v. \end{aligned} \tag{2.4.58}$$

Hence,

$$\mathbb{E}(S_{ut}^2) = \sigma^2(v-1) + (v-k){\sigma_b}^2 + \boldsymbol{\tau}'(R_d - \boldsymbol{r}_d\boldsymbol{r}_d'/n)\boldsymbol{\tau}. \tag{2.4.59}$$

Similarly, we can show that

(ii)

$$\begin{aligned}\mathbb{E}(S_{ub}^2) &= (b-1)(\sigma^2 + k\sigma_b^2) \\ &\quad +\boldsymbol{\tau}'(R_d - \boldsymbol{r}_d\boldsymbol{r}_d'/n)\boldsymbol{\tau} - \boldsymbol{\tau}'C_d\boldsymbol{\tau}. \qquad (2.4.60)\end{aligned}$$

(iii)

$$\begin{aligned}\mathbb{E}(S^2) &= \mathbb{E}(\boldsymbol{Y}'A_3\boldsymbol{Y}), \quad \text{where } A_3 = I_n - J_n/n \\ &= (n-1)\sigma^2 + k(b-1){\sigma_b}^2 \\ &\quad +\boldsymbol{\tau}'(R_d - \boldsymbol{r}_d\boldsymbol{r}_d'/n)\boldsymbol{\tau}. \qquad (2.4.61)\end{aligned}$$

(iv)

$$\mathbb{E}({S_t}^2) = \mathbb{E}(\boldsymbol{Q}'C_d^-\boldsymbol{Q}).$$

Under the inter-block model, we have seen that $\mathbb{E}(\boldsymbol{Q}) = C_d\boldsymbol{\tau}$ and $\mathbb{D}(\boldsymbol{Q}) = \sigma^2 C_d$. Therefore,

$$\begin{aligned}\mathbb{E}(\boldsymbol{Q}'C_d^-\boldsymbol{Q}) &= \boldsymbol{\tau}'C_dC_d^-C_d\boldsymbol{\tau} + \sigma^2\text{tr}(C_d^-C_d) \\ &= \boldsymbol{\tau}'C_d\boldsymbol{\tau} + \sigma^2\text{tr}(C_d^-C_d).\end{aligned}$$

Since $C_d^-C_d$ is an idempotent matrix, $\text{tr}(C_d^-C_d) = Rank(C_d^-C_d)$. But, firstly, $Rank(C_d^-C_d) \leq Rank(C_d)$ and secondly, $C_d = C_dC_d^-C_d$ gives $Rank(C_d) \leq Rank(C_d^-C_d)$, so that $Rank(C_d^-C_d) = Rank(C_d) = v-1$. Thus,

$$\mathbb{E}({S_t}^2) = \boldsymbol{\tau}'C_d\boldsymbol{\tau} + \sigma^2(v-1). \qquad (2.4.62)$$

(v)

$$\begin{aligned}\mathbb{E}({R_0}^2) &= \mathbb{E}(S^2) - \mathbb{E}(S_{ub}^2) - \mathbb{E}({S_t}^2) \\ &= (n-v-b+1)\sigma^2. \qquad (2.4.63)\end{aligned}$$

(vi)

$$\begin{aligned}\mathbb{E}({S_b}^2) &= \mathbb{E}(S^2) - \mathbb{E}(S_{ut}^2) - \mathbb{E}({R_0}^2) \\ &= (b-1)\sigma^2 + (bk-v){\sigma_b}^2. \qquad (2.4.64)\end{aligned}$$

From (2.4.63) and (2.4.64), we get unbiased estimators of σ^2 and ${\sigma_b}^2$ as

$$\hat{\sigma}^2 = {R_0}^2/(n-v-b+1) = {s_e}^2 \text{ say}; \qquad (2.4.65)$$

$$\begin{aligned}\hat{\sigma}_b{}^2 &= \{S_b{}^2 - (b-1)s_e{}^2\}/(n-v) \\ &= (b-1)(s_b{}^2 - s_e{}^2)/(n-v),\end{aligned} \tag{2.4.66}$$

where $s_b{}^2 = S_b{}^2/(b-1)$. We can now obtain estimates of the weights ω_1 and ω_2 as

$$\begin{aligned}\hat{\omega}_1 &= s_e^{-2}, \\ \hat{\omega}_2 &= \frac{n-v}{(n-k)s_b^2 - (v-k)s_e^2}.\end{aligned} \tag{2.4.67}$$

It may be noted that the estimator of σ_b^2 given by (2.4.66) could be negative for a given data set. In that case, we take $\hat{\sigma}^2$ as an estimator for σ_b^2.

Another alternative to the Yates procedure, outlined above for estimating the variance components σ^2 and σ_b^2 is based on the maximum likelihood method. For adopting this approach, we assume normality of observations. Recalling the inter-block model, we have seen that one can write the model as

$$\mathbb{E}(\boldsymbol{Y}) = A\boldsymbol{\theta}, \quad \mathbb{D}(\boldsymbol{Y}) = \sigma^2 V, \tag{2.4.68}$$

where

$$A = (\mathbf{1}_n \;\; D'_{1d}), \quad \boldsymbol{\theta}' = (\mu \;\; \boldsymbol{\tau}'), \; V = \delta D'_{2d}D_{2d} + I_n \text{ and } \delta = \sigma_b^2/\sigma^2. \tag{2.4.69}$$

Assume that $\boldsymbol{Y} \sim N_n(A\boldsymbol{\theta}, \sigma^2 V)$, i.e., the observations vector follows an n-variate normal distribution with mean vector $A\boldsymbol{\theta}$ and dispersion matrix $\sigma^2 V$. The logarithm of the likelihood function L for $\boldsymbol{Y}$ is then

$$-\frac{n}{2}\log_e 2\pi - \frac{n}{2}\log_e \sigma^2 - \frac{1}{2}\log_e \det(V) - \frac{1}{2\sigma^2}(\boldsymbol{Y} - A\boldsymbol{\theta})'V^{-1}(\boldsymbol{Y} - A\boldsymbol{\theta}). \tag{2.4.70}$$

It has been shown by Hartley and Rao (1967) that the maximum likelihood estimators of $\boldsymbol{\theta}$, δ and σ^2 can be obtained as solutions of the following equations:

$$\frac{1}{\sigma^2}A'V^{-1}(\boldsymbol{Y} - A\boldsymbol{\theta}) = \mathbf{0}, \tag{2.4.71}$$

$$-\frac{1}{2}\mathrm{tr}(V^{-1}D'_{2d}D_{2d}) + \frac{1}{2\sigma^2}(\boldsymbol{Y} - A\boldsymbol{\theta})'V^{-1}D'_{2d}D_{2d}V^{-1}(\boldsymbol{Y} - A\boldsymbol{\theta}) = 0, \tag{2.4.72}$$

$$-\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}(\boldsymbol{Y} - A\boldsymbol{\theta})'V^{-1}(\boldsymbol{Y} - A\boldsymbol{\theta}) = 0. \tag{2.4.73}$$

An advantage of this procedure is that the fixed effects $\boldsymbol{\tau}$ and the variance components are estimated simultaneously via an iterative process. For more details, we refer to Hartley and Rao (1967). For some related work on the method of maximum likelihood, see also Patterson and Thompson (1971) and Hemmerle and Hartley (1973).

Remark 2.4.1 It is clear that the estimators of treatment contrasts based on the combined intra-inter-block estimators will be affected when the weights ω_1 and ω_2 are estimated by substituting unbiased estimators of the variance components σ^2 and σ_b^2 in the expressions for $\omega_i, i = 1, 2$. The two questions that arise in this situation are: (i) is the combined estimator of a treatment contrast unbiased? (ii) does the combined estimator has a smaller variance than that of the intra-block estimator? Roy and Shah (1962) answered the first question in the affirmative for the Yates procedure. Furthermore, Shah (1964) showed that the combined estimator obtained through the Yates procedure has smaller variance than that of the corresponding intra-block estimator if $k\sigma_b^2 \leq \sigma^2$. However, this condition may often not hold. For reviews, additional references and other results on this topic, see Shah (1975, 1992) and Bhattacharya (1998).

2.5 Efficiency Factor

In practice, one may often be interested to know whether an incomplete block design is more efficient than a corresponding randomized *complete* block design or, more generally, in relation to an orthogonal design. One might also be interested to compare several incomplete block designs with respect to their efficiency. For making this assessment, the efficiency factor of a design can be used. Recall that we have introduced the term efficiency factor earlier in this chapter while discussing efficiency-balanced design as the harmonic mean of the canonical efficiency factors. Often, the efficiency factor of a design is computed by comparing the average variance of the BLUEs of elementary contrasts for the design under consideration with that for a comparable randomized complete block design (in case the design under consideration is equireplicate). For equireplicate designs, the efficiency factor defined in Section 2.3 equals the efficiency factor derived as the ratio of the average variance of the BLUEs of elementary treatment contrasts, as indicated above; however, for unequireplicated designs, these two might be different. See Pearce (1970) for a discussion on this aspect and also Kempthorne (1956).

In this section, we briefly introduce the notion of efficiency factor for incomplete block designs which are equireplicate. Throughout this section, only the intra-block model is considered.

Let d be a connected incomplete block design with v treatments. We first have the following result.

Lemma 2.5.1 *The average variance of the BLUEs of all elementary treatment contrasts under a connected incomplete block design d is inversely proportional to the harmonic mean of the positive eigenvalues of C_d, the C-matrix of d.*

Proof. Let U be a $\binom{v}{2} \times v$ matrix defined as

$$U = \begin{bmatrix} \mathbf{1}_{v-1} & & & & & -I_{v-1} \\ \mathbf{0}_{v-2} & \mathbf{1}_{v-2} & & & & -I_{v-2} \\ \mathbf{0}_{v-3} & \mathbf{0}_{v-3} & \mathbf{1}_{v-3} & & & -I_{v-3} \\ \vdots & & & & & \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \cdots & \mathbf{1}_2 & -I_2 \\ 0 & 0 & 0 & \cdots & 0 \quad 1 & -1 \end{bmatrix},$$

where, as before, for a positive integer s, $\mathbf{1}_s$ is an $s \times 1$ vector of all ones and $\mathbf{0}_s$ is an $s \times 1$ null vector. Clearly then, $U\boldsymbol{\tau}$ represents all the $\binom{v}{2}$ elementary treatment contrasts. If $\hat{\boldsymbol{\tau}}$ is a solution of the intra-block normal equations $C_d\boldsymbol{\tau} = \boldsymbol{Q}$, then the BLUE of $U\boldsymbol{\tau}$ is given by $U\hat{\boldsymbol{\tau}}$ and the dispersion matrix of $U\hat{\boldsymbol{\tau}}$ is $\mathbb{D}(U\hat{\boldsymbol{\tau}}) = \sigma^2 U C_d^- U'$, where C_d^- is an arbitrary g-inverse of C_d. Note that since $\mathcal{R}(U) \subseteq \mathcal{R}(C_d)$, the above dispersion matrix is invariant with respect to the choice of a g-inverse of C_d. The average variance of the BLUEs of all elementary treatment contrasts under the design d is then

$$\frac{\sigma^2}{\binom{v}{2}} \operatorname{tr}\left(U C_d^- U'\right) = \frac{\sigma^2}{\binom{v}{2}} \operatorname{tr}\left(U C_d^+ U'\right), \tag{2.5.1}$$

where C_d^+ is the Moore-Penrose inverse. Since d is connected, recall from Section 2.2 that the row sums of C_d^+ are all equal to zero. Now,

$$\begin{aligned} \operatorname{tr}\left(U C_d^+ U'\right) &= \operatorname{tr}\left(U' U C_d^+\right) \\ &= \operatorname{tr}\left[(vI_v - J_v) C_d^+\right] \\ &= \operatorname{tr}\left(v C_d^+\right) = v \sum_{i=1}^{v-1} \lambda_{di}^{-1}. \end{aligned} \tag{2.5.2}$$

where $\lambda_{d1}, \ldots, \lambda_{d,v-1}$ are the positive eigenvalues of C_d. Thus the average variance is

$$\frac{2\sigma^2}{v(v-1)} v \sum_{i=1}^{v-1} \lambda_{di}^{-1} = \frac{2\sigma^2}{H}, \tag{2.5.3}$$

where $H = \left(\frac{1}{v-1} \sum_{i=1}^{v-1} \lambda_{di}^{-1}\right)^{-1}$ is the harmonic mean of the positive eigenvalues of C_d. □

Note that the result of Lemma 2.5.1 holds for *any* connected design d, not necessarily equireplicate. Let d_1 be a randomized complete block design with v treatments and r replicates, where r is common replication number in d, now assumed to be equireplicate. The average variance of the BLUEs of all elementary contrasts under the design d_1 is $2\sigma_1^2/r$, where σ_1^2 is the per observation variance in the randomized complete block experiment. We can then define the efficiency of d relative to d_1 as the ratio

$$\text{Efficiency} = \frac{\text{av. var.}(\hat{\tau}_i - \hat{\tau}_j)_{d_1}}{\text{av. var.}(\hat{\tau}_i - \hat{\tau}_j)_d} = \frac{2\sigma_1^2/r}{2\sigma^2/H} = \left(\frac{\sigma_1^2}{\sigma^2}\right)(H/r). \tag{2.5.4}$$

The quantity $E = H/r$ is called the *efficiency factor* of the incomplete block design d. Though the efficiency factor of an incomplete block design is a useful measure of the efficiency of an incomplete block design, the actual efficiency depends on the intra-block variances σ^2 and σ_1^2. If one assumes that the intra-block variance is proportional to the block size, then it is expected that $\sigma_1^2 > \sigma^2$, as the design d has a block size which will be typically smaller than that in d_1.

Several upper bounds to the efficiency factor of incomplete block designs are available in the literature. Here we present one such bound for a proper, equireplicate incomplete block design d with v treatments, b blocks, common block size k and replication r. We also denote the incidence matrix of d by $N_d = (n_{dij})$. Denoting as before the positive eigenvalues of C_d by $\lambda_{d1}, \ldots, \lambda_{d,v-1}$, we have by the arithmetic mean-harmonic mean inequality

$$\begin{aligned}
(v-1)H &\leq \sum_{i=1}^{v-1} \lambda_{di} \\
&= \operatorname{tr}(C_d) \\
&= vr - k^{-1} \sum_{i=1}^{v} \sum_{j=1}^{b} n_{dij}^2
\end{aligned}$$

$$= \quad vr - k^{-1}vr, \tag{2.5.5}$$

the last step being the consequence of the fact that d is binary (i.e., $n_{dij} = 0, 1$). We thus have,

$$(v-1)H \leq \frac{k-1}{k}vr. \tag{2.5.6}$$

Recalling that $E = H/r$, we finally have

$$E \leq \frac{(k-1)v}{(v-1)k}. \tag{2.5.7}$$

Since for an incomplete block design d, $k < v$, we have $E < 1$. For various other results on the efficiency factor of incomplete block designs, including sharper upper bounds, see e.g., Connife and Stone (1974, 1975), Jarrett (1977, 1983, 1989), Jacroux (1984b), Paterson (1983) and Tjur (1990). Connife and Stone (1974) in particular obtained an upper bound of the efficiency factor for a binary, equireplicate proper design which is not balanced. For such a design, as before let v denote the number of treatments, r the common replication and k, the constant block size. Also, let λ_{ij} denote the number of times the treatments i and j appear together in a block of the design. Connife and Stone (1974) showed that a lower bound to the average variance of BLUEs of all elementary treatment contrasts is

$$V = 2\sigma^2 \left\{ \frac{v-2}{A-(v-1)^{\frac{1}{2}}(v-2)^{\frac{1}{2}}P} + \frac{1}{A+(v-1)^{\frac{1}{2}}(v-2)^{\frac{1}{2}}P} \right\}, \tag{2.5.8}$$

where $A = vr(k-1)/k$ and $P = k^{-1}\left\{\sum_{i \neq j}(\lambda_{ij} - \bar{\lambda})^2\right\}^{\frac{1}{2}}$, $\bar{\lambda}$ being the average of the λ_{ij}-values. It follows then that an upper bound for the efficiency factor, as defined earlier in this section is $2/(rV)$. Connife and Stone (1975) show that this upper bound is attained by a certain class of group divisible designs (such designs will be studied in greater detail in Chapter 4). For an elegant summary of the various upper bounds for the efficiency factor, a reference may be made to John and Williams (1995).

2.6 Exercises

In Exercises 2.1–2.23, assume the intra-block model.

2.1. Show that C_d, the C-matrix of a block design is nonnegative definite.

2.2. Show that for a connected block design d, the diagonal elements of C_d are all positive.

2.3. Provide a proof of the assertion in Remark 2.2.1. Also show that $\mathrm{pr}^{\perp}(L) = \mathrm{pr}^{\perp}(D'_{2d})$.

2.4. Consider an incomplete block design d with $v = 10, b = 3$ and block contents as given below:

$$(1,2,3,4); (1,5,6,7); (1,8,9,10).$$

Examine whether d is connected.

2.5. Consider the following incomplete block design with $v = 7$ treatments, $1, 2, \ldots, 7$ and $b = 7$ blocks:

$$(1,2,3); (1,2,4); (1,3,4); (2,3,4); (5,6); (5,7); (6,7).$$

Show that the design is disconnected and find a set of linearly independent treatment contrasts that are estimable under this design.

2.6. Consider a block design with v treatments and b blocks and let n be the total number of experimental units in the design. Show that a necessary condition for the design to be connected is that $n \geq b + v - 1$. Give an example to show that this condition is not sufficient for the connectedness of a block design.

2.7. Consider a connected block design d with v treatments and b blocks, such that $n = v + b - 1$, where n is the total number of experimental units in d. Show that d is necessarily binary.

2.8. Using the notations of Section 2.2, prove that for the data collected via a block design, the following identity holds:

$$S_t^2 + S_{ub}^2 = S_{ut}^2 + S_b^2.$$

2.9. Let $\boldsymbol{c}$ be a vector such that $C_d\boldsymbol{c} = \boldsymbol{p}$, where C_d is the C-matrix of a connected block design d. Show that the BLUE of $\boldsymbol{p}'\boldsymbol{\tau}$ is $\boldsymbol{c}'\boldsymbol{Q}$ and the variance of this estimator is $\sigma^2\boldsymbol{p}'\boldsymbol{c}$.

2.10. Show that for a connected block design d with v treatments, $C_d + aJ_v$, where $a \neq 0$ is a scalar, is nonsingular and $(C_d + aJ_v)^{-1}$ is a g-inverse of C_d.

2.11. Let d be a proper block design with v treatments, b blocks and block size k. If C_d is the C-matrix of d, show that $\mathrm{tr}(C_d) \leq b(k-1)$. When does the equality hold?

2.12. Give an example of an orthogonal block design whose incidence matrix is not a multiple of J_{vb}, where v is the number of treatments and b, the number of blocks.

2.13. Let $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_{v-1}$ be the eigenvalues of C_d, the C-matrix of a connected block design d. Show that the variance of the BLUE of an elementary contrast of treatment effects is bounded below by σ^2/λ_{v-1} and bounded above by σ^2/λ_1.

2.14. Consider a randomized complete block design with $v = 4$ treatments and $b = 5$ blocks and for $1 \leq i \leq 4$, let τ_i denote the effect of the ith treatment. Find the variances of and covariances between the BLUEs of the following treatment contrasts:

$$\tau_1 - \tau_2, \tau_1 + \tau_2 - 2\tau_3, \tau_1 + \tau_2 + \tau_3 - 3\tau_4.$$

2.15. A randomized complete block design with v treatments and r blocks was originally planned. However, due to an error, treatment label 1 was applied twice in the first block and treatment label 2 was not applied at all in this block. Describe a procedure for testing the hypothesis of equality of all treatment effects based on the data collected from the design.

2.16. Prove that the two definitions of variance-balanced designs given in Definitions 2.3.1 and 2.3.2 are equivalent.

2.17. Let d be an equireplicate, proper incomplete block design with v treatments, b blocks, replication r, block size k and incidence matrix $N_d = (n_{dij})$ such that $\sum_{j=1}^{b} n_{dij} n_{di'j} = \lambda$ for all $i \neq i', 1 \leq i, i' \leq v$.
Show that for such a design, $\sum_{j=1}^{b} n_{dij}^2 = rk - \lambda(v-1)$.

2.18. Consider an incomplete block design with $v = 5$ treatments, $b = 8$ blocks and block contents as follows:

$$(1,2,3); (1,2,4); (1,3,4); (2,3,4); (1,1,5); (2,2,5); (3,3,5); (4,4,5).$$

Examine whether this design is variance-balanced.

2.19. Prove that the matrix A_d given in (2.3.9) is nonnegative definite.

2.20. Consider an incomplete block design with $v = 5$ treatments, labeled $1, 2, \ldots, 5$ and $b = 12$ blocks, where the first 6 blocks have the following contents

$$(1,2); (1,3); (1,4); (2,3); (2,4); (3,4),$$

and the remaining 6 blocks are obtained by augmenting each of the above 6 blocks by two replications of the treatment with label 5. Examine whether the design is variance-balanced, efficiency-balanced or both.

2.21. Provide a proof of Lemma 2.3.1.

2.22. Show that a connected efficiency-balanced design has efficiency factor $\epsilon = 1$ if and only if the design is orthogonal.

2.23. Let $\boldsymbol{p}_1'\boldsymbol{\tau}, \boldsymbol{p}_2'\boldsymbol{\tau}, \ldots, \boldsymbol{p}_u'\boldsymbol{\tau}$ be a set of u treatment contrasts such that each of them is estimable under an incomplete block design d. Define $P = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_u)'$ and write $P\boldsymbol{\tau} = (\boldsymbol{p}_1'\boldsymbol{\tau}, \ldots, \boldsymbol{p}_u'\boldsymbol{\tau})'$. Under the usual notations, show that (a) $\mathbb{D}(P\hat{\boldsymbol{\tau}}) - \sigma^2 P R_d^{-1} P' \geq 0$ and, (b) $\mathbb{D}(P\hat{\boldsymbol{\tau}}) = \sigma^2 P R_d^{-1} P'$ if and only if $P R_d^{-1} N_d' = \mathbf{0}$ and in that case, $P\hat{\boldsymbol{\tau}} = P R_d^{-1} \boldsymbol{T}$.

2.24. Show that in the case of a randomized complete block design, no inter-block information is available.

2.25. Consider the combined estimation of a treatment contrast based on both intra- and inter-block information. If the best linear unbiased estimator of an estimable treatment contrast $\boldsymbol{p}'\boldsymbol{\tau}$ is given by $\boldsymbol{q}'(\omega_1 \boldsymbol{Q} + \omega_2 \boldsymbol{Q}^*)$ for some vector $\boldsymbol{q}$, find the variance of this estimator.

2.26. Recall the definition of ρ from Section 2.4 (equation (2.4.4)). Interpret the situations (i) $\rho^{-1} = 0$ and (ii) $\rho^{-1} = 1$.

2.27. Using the notations in Section 2.4 and writing $S_{ub}^2 = \boldsymbol{Y}' M \boldsymbol{Y}$, where $M = k^{-1} D_{2d}' D_{2d} - n^{-1} J_n$, prove the following:

(i) $M = (I_b - b^{-1} J_b) \otimes (k^{-1} J_k)$,

(ii) $M \mathbf{1}_n = \mathbf{0}$,

(iii) $M\Sigma = (\sigma^2 + k\sigma_b^2)(I_b - b^{-1} J_b) \otimes (k^{-1} J_k)$,

(iv) $D_{1d} M D_{1d}' = k^{-1} N_d N_d' - n^{-1} \boldsymbol{r}_d \boldsymbol{r}_d'$.

Using the above facts, prove the expression in (2.4.60).

Chapter 3

Balanced Designs

3.1 Introduction

Two notions of balance, viz., variance- and efficiency-balance were introduced in Section 2.3. In this chapter, a more detailed description of incomplete block designs that are balanced is provided. Among the variance-balanced designs, the most important ones are the balanced incomplete block (BIB) designs. In Section 3.2, we discuss some general properties of BIB designs. The methods of analysis described in Chapter 2 are specialized for the case of BIB designs in Section 3.3. In Section 3.4, some main methods of construction of BIB designs and a few results on the existence of such designs are discussed. Some generalizations of BIB designs are considered in Section 3.5. In Section 3.6, we describe some construction methods of variance-balanced designs that are not BIB designs as also of some families of efficiency-balanced designs. Finally, in Section 3.7, a brief description of nested balanced incomplete block designs is provided.

3.2 Some Properties of BIB Designs

We have described in Chapter 2 a BIB design through its incidence matrix. A formal definition of BIB designs follows.

Definition 3.2.1 *A balanced incomplete block (BIB) design is an arrangement of v treatments in b blocks such that*
(i) each block contains $k(< v)$ distinct treatments,
(ii) each treatment appears in r blocks,
(iii) each pair of treatments appears together in λ blocks.

Thus, v is the number of treatments, b is the number of blocks, r is the replication of each treatment, the block size is k and the parameter λ is sometimes called the *pairwise concurrence parameter* or simply the *concurrence parameter.*

Example 3.2.1 A BIB design with $v = 7 = b, r = 3 = k, \lambda = 1$ is shown below where the treatments are labeled as $1, 2, \ldots, 7$.

Block	Block contents
I	(1,2,4)
II	(2,3,5)
III	(3,4,6)
IV	(4,5,7)
V	(1,5,6)
VI	(2,6,7)
VII	(1,3,7)

BIB designs are the most important among the incomplete block designs and have been studied and used extensively. Many examples of the use of BIB designs in planning actual experiments in diverse areas can be found in standard texts, e.g., Cochran and Cox (1957), Cox (1958) and Wu and Hamada (2000). In recent years, the use of incomplete block designs including BIB designs has been made in some other areas, for example, in deriving 'optimal' visual cryptographic schemes; see e.g., Bose and Mukerjee (2006) and the references given therein.

From the results of Chapter 2, it is clear that a BIB design is variance-balanced. As we shall see in Chapter 6, these designs have strong optimality properties and thus are important from a statistical perspective. Also, there are many challenging and interesting problems related to the construction and existence of these designs and are therefore of interest to combinatorial mathematicians as well.

Historically, BIB designs were introduced in the statistics literature by Yates (1936a). However, combinatorial structures which we now recognize as BIB designs were known even in the 19th century. Kirkman (1850) solved the following problem, originally proposed by Woolhouse (1844):

A school mistress is in the habit of taking 15 girls of her school for a daily morning walk in 5 batches of 3 girls each, so that each girl has 2 companions. Is it possible to find an arrangement so that for 7 consecutive days, no girl walks with any of her companions in any batch more than once?

The solution of the above problem (called the Kirkman's schoolgirl problem) has a one-one correspondence with the solution of a BIB design and such a BIB design is also called a Kirkman Triple System, KTS(15). A KTS(15) is shown below, where the schoolgirls are labeled $1, 2, \ldots, 15$:

Day 1	Day 2	Day 3	Day 4	Day 5	Day 6	Day 7
1,6,11	1,8,10	1,3,9	1,2,5	2,3,6	1,7,14	1,12,13
2,7,12	2,9,11	2,13,14	3,10,12	5,7,13	3,5,11	2,4,10
3,8,13	3,4,7	4,5,8	4,11,13	8,9,12	4,6,12	5,6,9
4,9,14	5,12,14	6,7,10	6,8,14	10,11,14	9,10,13	7,8,11
5,10,15	6,13,15	11,12,15	7,9,15	1,4,15	2,8,15	3,14,15

It is easily seen that the above plan is a BIB design with parameters $v = 15, b = 35, r = 7, k = 3, \lambda = 1$ when triplets of girls are treated as blocks. Other solutions to KTS(15) were provided by several authors, including Cayley (1850), Peirce (1860) and Davis (1897). The solution of a Kirkman Triple System KTS(m) for all $m \equiv 3 \pmod 6$ was provided by Raychaudhuri and Wilson (1971).

Steiner (1853) proposed the problem of arranging n objects in triplets (called Steiner's triple systems) such that every pair of objects appears in exactly one triplet. It is easy to see that Steiner's triples are in fact BIB designs with block size three. Early important contributions in respect of BIB designs were made by Bose (1939, 1942, 1949), Fisher (1940) and Yates (1940). For an elegant description of the early history of combinatorial designs, including BIB designs, see Anderson, Colbourn, Dinitz and Griggs (2007).

The integers v, b, r, k, λ are called the *parameters* of a BIB design. Throughout, we take $k \geq 2$. The parameters of a BIB design are related by the following identities:

$$\begin{aligned} vr &= bk, \\ \lambda(v-1) &= r(k-1). \end{aligned} \tag{3.2.1}$$

The first identity in (3.2.1) is trivially true as each side represents the total number of experimental units in the design. To see the truth of the second identity, we proceed as follows: Let N_d be the $v \times b$ incidence matrix of a BIB design d with parameters v, b, r, k, λ. Then, N_d is a matrix with entries zero and 1 and the following facts follow from Definition 3.2.1:

$$N_d \mathbf{1}_b = r\mathbf{1}_v, \quad \mathbf{1}_v' N_d = k\mathbf{1}_b', \quad N_d N_d' = (r-\lambda)I_v + \lambda J_v. \tag{3.2.2}$$

By the third relation in (3.2.2) we have

$$N_dN_d'\mathbf{1}_v = (r-\lambda)\mathbf{1}_v + v\lambda\mathbf{1}_v = \{r+\lambda(v-1)\}\mathbf{1}_v. \tag{3.2.3}$$

Also,

$$N_dN_d'\mathbf{1}_v = N_d(N_d'\mathbf{1}_v) = kN_d\mathbf{1}_b = rk\mathbf{1}_v. \tag{3.2.4}$$

The result now follows by comparing the right sides of (3.2.3) and (3.2.4).

We have already seen in Theorem 2.3.3 that the Fisher's inequality holds for a wide class of variance-balanced designs. Since BIB designs are also variance-balanced and equireplicate, the inequality $b \geq v$ holds for all BIB designs as well. Here we give a (direct) proof of this inequality in the context of BIB designs. As before, let N_d denote the incidence matrix of a BIB design d with parameters v, b, r, k, λ. Then, from (3.2.2) it is easy to see that

$$\det(N_dN_d') = (r-\lambda)^{v-1}\{r+\lambda(v-1)\} = rk(r-\lambda)^{v-1}, \tag{3.2.5}$$

by virtue of the second identity in (3.2.1). Since $r > \lambda$, we conclude that the square matrix N_dN_d' of order v is nonsingular. Therefore,

$$v = Rank(N_dN_d') = Rank(N_d) \leq b. \tag{3.2.6}$$

The relations $vr = bk, r(k-1) = \lambda(v-1)$ and $b \geq v$ are only necessary but not sufficient for the existence of a BIB design, except in some special cases. That is, given integers v, b, r, k, λ satisfying the above three conditions, it may not be possible to construct a BIB design with these parameters. We shall elaborate on this point subsequently in this chapter.

Given a BIB design, one can get another BIB design through complementation. Let d be a BIB design with parameters (v, b, r, k, λ) and let $\bar{d}$ be its *complementary* design, obtained by including in the jth block of $\bar{d}$ all those treatments which do not appear in the jth block of d, $1 \leq j \leq b$. Let $\bar{v}$ and $\bar{b}$ respectively, denote the number of treatments and blocks in $\bar{d}$. Then trivially, $\bar{v} = v, \bar{b} = b$. Consider an arbitrary treatment α. Since this treatment appears in r blocks of d, it appears in precisely $\bar{r} = b - r$ blocks of $\bar{d}$. Similarly, the block size of $\bar{d}$ is $\bar{k} = v - k$. Now consider an arbitrary pair of treatments α, β. This pair appears together in λ blocks of d. Furthermore, there are precisely $(r-\lambda)$ blocks in d which contain α but not β and another $(r-\lambda)$ blocks in d which contain β but not α. Thus, there are exactly $\bar{\lambda} = b - 2r + \lambda$ blocks in d which contain neither α nor β. These number of blocks in $\bar{d}$ contain both α and β, which shows that $\bar{d}$ is a BIB design with parameters $\bar{v}, \bar{b}, \bar{r}, \bar{k}, \bar{\lambda}$.

A BIB design with parameters v, b, r, k, λ is called *symmetric* if $b = v$. The next result gives an important property of symmetric BIB designs.

Theorem 3.2.1 *Any pair of distinct blocks of a symmetric balanced incomplete block design has exactly λ treatments in common.*

Proof. Let d be a symmetric BIB design with parameters $v = b, r = k, \lambda$. Then clearly, N_d is a *square* matrix of order v and

$$N_d \mathbf{1}_v = r\mathbf{1}_v = N_d' \mathbf{1}_v, \quad \mathbf{1}_v' N_d = r\mathbf{1}_v' = \mathbf{1}_v' N_d'.$$

Hence,

$$\begin{aligned} N_d' N_d N_d' &= (r-\lambda)N_d' + \lambda N_d' \mathbf{1}_v \mathbf{1}_v' \\ &= (r-\lambda)N_d' + \lambda r \mathbf{1}_v \mathbf{1}_v' \\ &= (r-\lambda)N_d' + \lambda \mathbf{1}_v (r\mathbf{1}_v') \\ &= (r-\lambda)N_d' + \lambda \mathbf{1}_v \mathbf{1}_v' N_d'. \end{aligned} \tag{3.2.7}$$

Also, we have seen earlier that $v = Rank(N_d N_d') = Rank(N_d')$, which means that N_d' is invertible. Postmultiplying both sides of (3.2.7) by $(N_d')^{-1}$, one obtains

$$N_d' N_d = (r-\lambda)I_v + \lambda J_v, \tag{3.2.8}$$

which shows that each off-diagonal element of $N_d' N_d$ equals λ. The off-diagonal element in the uth row and tth column ($u \neq t$) of $N_d' N_d$ also equals the inner product of the uth and tth columns of N_d and since this inner product is precisely the number of treatments common between the uth and tth blocks of d, the proof is complete. □

We next show that the existence of a *symmetric* BIB design implies the existence of two more BIB designs. Suppose d is a symmetric BIB design with parameters v, k, λ. Choose a block of d and delete from d the chosen block and all the treatments contained in it. Call the design consisting of the remaining structure, d_1. Clearly d_1 has $b_1 = v - 1$ blocks. Also, it is easy to see that d_1 has $v_1 = v - k$ treatments. Since by Theorem 3.2.1, the number of treatments common between the deleted block and each of the remaining blocks is λ, each block of d_1 has $k_1 = k - \lambda$ distinct treatments. The replication of treatments and pairs of treatments not appearing in the deleted block remain unaltered and so, in d_1, each treatment appears in $r_1 = k$ blocks and each pair of treatment appears together in $\lambda_1 = \lambda$ blocks. It follows then that d_1 is a BIB design with parameters

$$v_1 = v - k, b_1 = v - 1, r_1 = k, k_1 = k - \lambda, \lambda_1 = \lambda. \tag{3.2.9}$$

The above procedure of obtaining a BIB design from a symmetric BIB design has been called the process of *block section* by Bose (1939) and d_1 is called the *residual design* of the symmetric BIB design d.

Next, starting from a symmetric BIB design d with parameters v, k, λ, one can delete a block of d and retain only those treatments in the remaining blocks which appear in the deleted block. Let the resultant design be denoted by d_2. The number of treatments in d_2 is obviously $v_2 = k$ and the number of blocks is $b_2 = v - 1$. Since the number of treatments common between the deleted block and each of the remaining blocks is λ, the block size of d_2 is $k_2 = \lambda$. Also, since each treatment in d_2 appears once in the deleted block, each treatment appears in $r_2 = k - 1$ blocks of d_2. A similar argument shows that any pair of treatments appears together in $\lambda_2 = \lambda - 1$ blocks of d_2. We therefore conclude that d_2 is a BIB design with parameters

$$v_2 = k, b_2 = v - 1, r_2 = k - 1, k_2 = \lambda, \lambda_2 = \lambda - 1. \tag{3.2.10}$$

The above procedure of obtaining a BIB design d_2 from a symmetric BIB design has been called the process of *block intersection* by Bose (1939) and d_2 is called the *derived design* of the symmetric BIB design d.

To illustrate the above ideas, let us consider a symmetric BIB design with parameters $v = 11 = b, r = 6 = k, \lambda = 3$, a solution of which is given below:

$$(\mathbf{2}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{10}, \mathbf{11}), (1, 3, \mathbf{7}, \mathbf{8}, 9, \mathbf{11}), (1, \mathbf{2}, 4, \mathbf{8}, 9, \mathbf{10}),$$
$$(\mathbf{2}, 3, 5, 9, \mathbf{10}, \mathbf{11}), (1, 3, 4, \mathbf{6}, \mathbf{10}, \mathbf{11}), (1, \mathbf{2}, 4, 5, \mathbf{7}, \mathbf{11}),$$
$$(1, \mathbf{2}, 3, 5, \mathbf{6}, \mathbf{8}), (\mathbf{2}, 3, 4, \mathbf{6}, \mathbf{7}, 9), (3, 4, 5, \mathbf{7}, \mathbf{8}, \mathbf{10}),$$
$$(4, 5, \mathbf{6}, \mathbf{8}, 9, \mathbf{11}), (1, 5, \mathbf{6}, \mathbf{7}, 9, \mathbf{10}).$$

Let us choose the first block as the one that is to be deleted. The deleted treatments are shown in bold face in all the blocks. The residual design then has the following contents:

$$(1, 3, 9), (1, 4, 9), (3, 5, 9), (1, 3, 4), (1, 4, 5),$$

$$(1,3,5),(3,4,9),(3,4,5),(4,5,9),(1,5,9).$$

It can be verified easily that the above is a BIB design with parameters $v_1 = 5, b_1 = 10, r_1 = 6, k_1 = 3, \lambda_1 = 3$ and involves treatment labels 1,3,4,5,9. Similarly, the derived design has the following blocks:

$$(7,8,11),(2,8,10),(2,10,11),(6,10,11),(2,7,11),$$

$$(2,6,8),(2,6,7),(7,8,10),(6,8,11),(6,7,10).$$

This is again a BIB design with parameters $v_2 = 6, b_2 = 10, r_2 = 5, k_2 = 3, \lambda_2 = 2$ involving treatment labels 2,6,7,8,10,11.

The Fisher's inequality $b \geq v$ for an arbitrary BIB design can be sharpened in some special cases. The next result shows this. Henceforth, for positive integers a, b, we shall write $a|b$ to mean that a divides b, i.e., $b \equiv 0 \pmod{a}$.

Theorem 3.2.2 *Consider a BIB design with parameters v, b, r, k, λ such that $r|b$. Then for such a design, the inequality*

$$b \geq v + r - 1 \tag{3.2.11}$$

holds.

Proof. Let d be a BIB design with parameters v, b, r, k, λ such that $r|b$ and let $b = nr$ where $n \geq 2$ is an integer. From the second identity in (3.2.1), we have

$$\begin{aligned} r &= \frac{\lambda(nk-1)}{k-1} \\ &= \frac{\lambda(n-1)}{k-1} + \lambda n. \end{aligned}$$

It follows then that $\lambda(n-1)/(k-1)$ is a positive integer. If possible, let $b < v + r - 1$. This implies that $nr < v + r - 1$, which on simplification leads to

$$\frac{\lambda(n-1)}{k-1} < 1.$$

This contradicts the above observed fact that $\lambda(n-1)/(k-1)$ is a positive integer. Hence the result. □

Remark 3.2.1 The inequality (3.2.11) was proved by Bose (1942) in the context of a subclass of BIB designs, called *resolvable* BIB designs; for such designs, a necessary condition is that b is divisible by r. However, as seen above, (3.2.11) holds for any BIB design with r dividing b and not merely for resolvable BIB designs.

We now formally introduce the notion of resolvable BIB design.

Definition 3.2.2 *A balanced incomplete block design with parameters v, b, r, k, λ is said to be resolvable if its blocks can be partitioned into r sets of blocks, each set containing b/r blocks such that every set contains each treatment precisely once. A resolvable BIB design is called affine resolvable if any two blocks belonging to two different sets intersect in a constant number of treatments.*

Example 3.2.2 Consider a BIB design with parameters $v = 9, b = 12, r = 4, k = 3, \lambda = 1$. A solution for this design is given below. This solution can easily be seen to be resolvable, as each treatment appears precisely once in each set (replication). Furthermore, it can be verified that the solution is actually affine resolvable.

Replication No.	Block No.	Block contents
	1.	(1,2,3)
I	2.	(4,5,6)
	3.	(7,8,9)
	4.	(1,4,7)
II	5.	(2,5,8)
	6.	(3,6,9)
	7.	(1,6,8)
III	8.	(2,4,9)
	9.	(3,5,7)
	10.	(1,5,9)
IV	11.	(2,6,7)
	12.	(3,4,8)

A solution of the Kirkman's schoolgirl problem was given in Section 3.1. It can be seen that this solution is also a resolvable BIB design, each day forming a resolvable set.

As observed earlier, (3.2.11) holds for any resolvable BIB design. In what follows, we give an alternative proof of (3.2.11) in the context of resolvable BIB designs. Let d be a resolvable BIB design with parameters v, b, r, k, λ and as before, let $n = b/r = v/k$. Since d is resolvable, its blocks can be partitioned into r sets, say $\{T_0\}, \{T_1\}, \ldots, \{T_{r-1}\}$, each set containing n blocks. For $0 \leq i \leq r-1$, let the blocks of the set $\{T_i\}$ be labeled as $B_{i1}, \ldots, B_{in}$. Consider the block $B_{01} \in \{T_0\}$ and let x_{ij} be the number of treatments common between B_{01} and B_{ij}, $1 \leq i \leq r-1, 1 \leq j \leq n$. If a treatment belongs to B_{01} then it cannot appear in any other block in $\{T_0\}$. Hence, it must appear $r-1$ times among the blocks in $\{T_1\}, \ldots, \{T_{r-1}\}$ and thus,

$$\sum_{i=1}^{r-1}\sum_{j=1}^{n} x_{ij} = k(r-1). \tag{3.2.12}$$

The average of the $\{x_{ij}\}$ is then

$$\bar{x} = \frac{1}{n(r-1)}\sum_{i=1}^{r-1}\sum_{j=1}^{n} x_{ij} = k/n = k^2/v. \tag{3.2.13}$$

Also, the $k(k-1)/2$ pairs of treatments in the block B_{01} each appear $\lambda - 1$ times in the sets $\{T_1\}, \ldots, \{T_{r-1}\}$ and we have

$$\sum_{i=1}^{r-1}\sum_{j=1}^{n} x_{ij}(x_{ij}-1)/2 = (\lambda-1)k(k-1)/2. \tag{3.2.14}$$

It follows then that

$$\begin{aligned}
\sum_{i,j} x_{ij}^2 &= \sum_{i,j} x_{ij}(x_{ij}-1) + \sum_{i,j} x_{ij} \\
&= (\lambda-1)k(k-1) + k(r-1) \\
&= k\{(r-1) + (\lambda-1)(k-1)\}.
\end{aligned} \tag{3.2.15}$$

Using the basic identity $\lambda = r(k-1)/(v-1) = r(k-1)/(nk-1)$, we have

$$\sum_{i,j} x_{ij}^2 = \frac{k\{(nk-1)(r-k) + r(k-1)^2\}}{nk-1}. \tag{3.2.16}$$

Let σ^2 denote the variance of the $\{x_{ij}\}$ values. Then,

$$\begin{aligned}
\sigma^2 &= \frac{\sum_{i,j}(x_{ij}-\bar{x})^2}{n(r-1)} \\
&= \frac{k\{(nk-1)(r-k) + r(k-1)^2\}}{n(r-1)(nk-1)} - \frac{k^2}{n^2} \\
&= \frac{k^2(n-1)\{r(n-1) - (nk-1)\}}{n^2(r-1)(nk-1)} \\
&= \frac{k(v-k)(b-r-v+1)}{n^2(r-1)(v-1)}.
\end{aligned}$$

Since $\sigma^2 \geq 0$, we have the inequality $b \geq v + r - 1$. If $b = v + r - 1$, then $\sigma^2 = 0$ and in such a case, each $x_{ij} = \bar{x} = k^2/v$. This implies that if $b = v + r - 1$, the block B_{01} has exactly k^2/v treatments in common with each of the blocks in the sets $\{T_i\}$, $1 \leq i \leq r - 1$. Since the choice of the block B_{01} was arbitrary, it follows that under the condition $b = v + r - 1$, any two blocks belonging to different sets have k^2/v treatments in common, i.e., the design is affine resolvable. Also, k^2/v must be an integer in such a case.

Conversely, if the BIB design is affine resolvable, x_{ij} is a constant $(= k^2/v)$ for all i, j, $1 \leq i \leq r-1, 1 \leq j \leq n$ and hence, $\sigma^2 = 0 \Rightarrow b = v + r - 1$. Combining all of the above, we have the following result.

Theorem 3.2.3 *If a BIB design with parameters v, b, r, k, λ is resolvable then*

$$b \geq v + r - 1. \tag{3.2.17}$$

Furthermore, if the design is affine resolvable, equality holds in (3.2.17). Conversely, if for a resolvable BIB design, equality in (3.2.17) holds, the design must be affine resolvable and in that case, k^2/v must be an integer, this being the number of treatments common between any two blocks belonging to different resolvable sets.

To see the kind of information contained in Theorem 3.2.3, first consider the BIB design with parameters $v = 9, b = 12, r = 4, k = 3, \lambda = 1$. We have already seen in Example 3.2.2 that this design has a resolvable solution. Also, since the parameters of this design satisfy the condition $b = v + r - 1$, the resolvable solution must in fact be affine resolvable. This confirms the earlier observed fact that the solution in Example 3.2.2 is indeed affine resolvable. Also, here $k^2/v = 1$. Again, consider a BIB design with parameters $v = 10, b = 18, r = 9, k = 5, \lambda = 4$. Here $r|b$ and $b = v + r - 1$. Thus, if this design were to have a resolvable solution, it must be affine resolvable, as per Theorem 3.2.3. However, in that case, k^2/v must be integral, which is not the case with the design under consideration. Hence, no solution of this BIB design can be resolvable.

The next result concerns the parameters of an affine resolvable BIB design.

Theorem 3.2.4 *The parameters of an affine resolvable BIB design can be expressed in terms of two nonnegative integers.*

Proof. Let d be an affine resolvable BIB design with parameters v, b, r, k, λ. Let $m = k^2/v$ (an integer), so that $k = mn$ where $n = b/r$. We then have

$$v = mn^2, b = nr, k = mn.$$

From Theorem 3.2.3, we have

$$nr = mn^2 + r - 1 \Rightarrow r = (mn^2 - 1)/(n-1) = mn + m + (m-1)/(n-1).$$

Also,

$$\begin{aligned}\lambda &= r(mn-1)/(mn^2-1)\\ &= (mn-1)/(n-1)\\ &= m + (m-1)/(n-1).\end{aligned}$$

Since λ is an integer, we must have

$$m = (n-1)u + 1$$

for some integer $u \geq 0$. It follows then that the parameters of an affine resolvable BIB design can be expressed as

$$\begin{array}{llll} v = & n^2\{(n-1)u+1\}, & b = & n(n^2u+n+1),\\ r = & n^2u+n+1, & k = & n\{(n-1)u+1\},\\ \lambda = & nu+1. & & \end{array}$$

□

A table of known resolvable BIB designs with $v \leq 100$ and $r \leq 10$ is given in Caliński and Kageyama (2000, Table 9.1).

Remark 3.2.2 The notion of resolvable incomplete block designs, including BIB designs, has been generalized to α-resolvable designs by Shrikhande and Raghavarao (1963) as follows: an equireplicate, proper incomplete block design with v treatments, b blocks, replication r and block size k is called α-resolvable if the blocks can be partitioned into t sets, say $\{T_1\}, \ldots, \{T_t\}$, each set containing β blocks, such that in each set $\{T_i\}, 1 \leq i \leq t$, every treatment appears $\alpha(\geq 1)$ times. For $\alpha = 1$, one gets the usual resolvable designs. For more details on α-resolvable designs, including construction methods, a reference may be made to Shrikhande and Raghavarao (1963).

3.3 Analysis of BIB Designs

In this section, we briefly discuss the analysis of BIB designs, specializing the results from Chapter 2. Let d be a BIB design with parameters v, b, r, k, λ and incidence matrix N_d. Note that a BIB design is connected whenever $k \geq 2$. It is easy to see that the C-matrix of d is given by

$$\begin{aligned} C_d &= rI_v - k^{-1}\{(r-\lambda)I_v + \lambda J_v\} \\ &= (\lambda v/k)(I_v - v^{-1}J_v). \end{aligned} \tag{3.3.1}$$

Then, the positive eigenvalues of C_d given by (3.3.1) are the same ($= \lambda v/k$) and a g-inverse of C_d is

$$C_d^- = (k/\lambda v)I_v. \tag{3.3.2}$$

Thus, a solution of the intra-block normal equations for estimating linear functions of treatment effects is

$$\hat{\boldsymbol{\tau}} = (k/\lambda v)\boldsymbol{Q} \tag{3.3.3}$$

where $\boldsymbol{Q} = (Q_1, \ldots, Q_v)'$ is the vector of adjusted treatment totals. Note that for $1 \leq i \leq v$,

$$Q_i = T_i - k^{-1}\sum_{j(i)} B_j, \tag{3.3.4}$$

where for $1 \leq i \leq v$ and $1 \leq j \leq b$, T_i is the total of observations from the ith treatment, B_j is the total of observations in the jth block and $\sum_{j(i)}$ denotes the sum over all those blocks which contain the ith treatment. It follows now that that under the intra-block model, the numerator sum of squares of the statistic $\mathcal{F}$ (*cf.* (2.2.28)) (also called the adjusted treatment sum of squares) for testing the hypothesis $H_0 : \tau_1 = \cdots = \tau_v$ is given by

$$\hat{\boldsymbol{\tau}}'\boldsymbol{Q} = (k/\lambda v)\boldsymbol{Q}'\boldsymbol{Q} = (k/\lambda v)\sum_{i=1}^{v} Q_i^2. \tag{3.3.5}$$

The (unadjusted) block sum of squares is as usual given by $k^{-1}\sum_{j=1}^{b} B_j^2 - G^2/bk$, where $G = \sum_{j=1}^{b} B_j$ is the grand total of all observations. The intra-block analysis of variance table can now be formed as below.

TABLE 3.3.1: Intra-block Analysis of Variance of BIB Designs

Source	d.f.	Sum of Squares
Treatments (adjusted)	$v-1$	$(k/\lambda v)\sum_i Q_i^2$
Blocks (unadjusted)	$b-1$	$\sum_j B_j^2/k - G^2/bk$
Intra-block Error	$bk-v-b+1$	By subtraction
Total	$bk-1$	$\sum_{i,j} Y_{ij}^2 - G^2/bk$

The test of the hypothesis of equality of all treatment effects can now be completed easily.

In Chapter 2 (*cf.* (2.5.7)), we have seen that the efficiency factor E of an incomplete block design with v treatments and constant block size k is bounded above by $\frac{(k-1)v}{(v-1)k}$. It is not hard to see that this upper bound is attained by a BIB design and in that sense, one can regard a BIB design as the most efficient incomplete block design. Observe that for a BIB design $E = (k-1)v/\{(v-1)k\} = \lambda v/rk$.

For a BIB design d with parameters v, b, r, k, λ, as observed earlier, the incidence matrix N_d satisfies

$$N_d N_d' = (r-\lambda)I_v + \lambda J_v,$$

which is nonsingular. Therefore,

$$(N_d N_d')^{-1} = \frac{1}{(r-\lambda)} I_v - \frac{\lambda}{rk(r-\lambda)} J_v. \tag{3.3.6}$$

From (2.4.14), the inter-block estimator of a treatment contrast $\boldsymbol{p}'\boldsymbol{\tau} = \sum_{i=1}^{v} p_i \tau_i$ is given by

$$\boldsymbol{p}'(N_d N_d')^{-1} N_d \boldsymbol{B} = (r-\lambda)^{-1} \boldsymbol{p}' N_d \boldsymbol{B} = \sum_{i=1}^{v} p_i B_i^*, \tag{3.3.7}$$

where for $1 \le i \le v$, $B_i^* = \sum_{j(i)} B_j$. Also, the intra-block estimator of $\boldsymbol{p}'\boldsymbol{\tau}$ is given by $\boldsymbol{p}' C_d^- \boldsymbol{Q} = (k/\lambda v)\boldsymbol{p}'\boldsymbol{Q}$. The combined intra-inter-block estimator can now be obtained, as indicated in Chapter 2. The weights ϕ_1 and ϕ_2 for a BIB design simplify to

$$\phi_1 = (\boldsymbol{p}'\boldsymbol{p})^{-1}(\lambda v)/(k\sigma^2),\ \phi_2 = (\boldsymbol{p}'\boldsymbol{p})^{-1}(r-\lambda)/\{k(k\sigma_b^2 + \sigma^2)\}. \tag{3.3.8}$$

Estimates of the weights ϕ_1, ϕ_2 can be obtained as indicated in Chapter 2. Specifically, an estimator of ϕ_1 can be obtained by using the error mean square from the intra-block analysis of variance table (Table 3.3.1) as an unbiased estimator of σ^2 and is thus given by

$$\hat{\phi}_1 = (p'p)^{-1}(\lambda v/k)/E_e, \tag{3.3.9}$$

where E_e is the error mean square from Table 3.3.1. For estimating ϕ_2, we first obtain an estimator of σ_b^2. Let S_b^2 denote the *adjusted* block sum of squares. Then,

$$\begin{aligned} S_b^2 &= \text{Treatment S.S. (adjusted)+ Block S.S. (unadjusted)} \\ &\quad -\text{Treatment S.S. (unadjusted)} \\ &= (k/\lambda v)\sum_{i=1}^{v} Q_i^2 + \sum_{j=1}^{b} B_j^2/k - \sum_{i=1}^{v} T_i^2/r. \end{aligned} \tag{3.3.10}$$

From (2.4.64),

$$\mathbb{E}(S_b^2) = (bk - v)\sigma_b^2 + (b-1)\sigma^2. \tag{3.3.11}$$

It follows then that an unbiased estimator of σ_b^2 is given by

$$\hat{\sigma_b}^2 = (E_b - E_e)(b-1)/(bk-v) = (E_b - E_e)(b-1)/\{v(r-1)\}, \tag{3.3.12}$$

where $E_b = S_b^2/(b-1)$. One can now easily get an estimator of ϕ_2. The variance of the BLUE of an elementary contrast, using both intra- and inter-block information is given by

$$\frac{2k}{\lambda v \sigma^{-2} + (r-\lambda)(k\sigma_b^2 + \sigma^2)^{-1}}. \tag{3.3.13}$$

As in (2.4.23), letting $\omega_1 = \sigma^{-2}$ and $\omega_2 = (k\sigma_b^2 + \sigma^2)^{-1}$, the variance of the BLUE of an elementary treatment contrast given by (3.3.13) can be written as

$$\begin{aligned} \frac{2k}{\lambda v \omega_1 + (r-\lambda)\omega_2} &= \frac{2k(v-1)}{\omega_1 \lambda v(v-1) + \omega_2(r-\lambda)(v-1)} \\ &= \frac{2}{r}\left\{\frac{k(v-1)}{v(k-1)\omega_1 + (v-k)\omega_2}\right\}. \end{aligned} \tag{3.3.14}$$

The quantity within the parenthesis on the right side of (3.3.14) is called the *effective variance* (see e.g., Chakrabarti (1962)) and denoted by σ_E^2, that is,

$$\sigma_E^2 = \frac{k(v-1)}{v(k-1)\omega_1 + (v-k)\omega_2}. \tag{3.3.15}$$

Since $\sigma_E^2 = \sigma^2\{1 + (v-k)\omega\}$, where

$$\omega = \frac{\omega_1 - \omega_2}{v(k-1)\omega_1 + (v-k)\omega_2}, \tag{3.3.16}$$

the effective variance can be approximated by

$$S_E^2 = E_e\{1 + (v-k)\omega\}. \tag{3.3.17}$$

For $1 \le i \le v$, let

$$W_i = (v-k)T_i - (v-1)\sum_{j(i)} B_j + (k-1)G \tag{3.3.18}$$

and

$$U_i = T_i + \omega W_i. \tag{3.3.19}$$

The quantities $\{U_i\}$ are sometimes called the adjusted treatment totals using both intra- and inter-block information. Let

$$S_U^2 = \sum_{i=1}^{v} U_i^2 - v^{-1}\left(\sum_{i=1}^{v} U_i\right)^2 \tag{3.3.20}$$

denote the corrected sum of squares due to the U_i's. The statistic

$$\mathcal{F}' = \frac{S_U^2}{(v-1)rE_e\{1 + (v-k)\hat{\omega}\}}, \tag{3.3.21}$$

under the hypothesis that all treatment effects are equal, has approximately an F-distribution on $(v-1)$ and $n_e = bk - v - b + 1$ degrees of freedom, where $\hat{\omega}$ is an estimate of ω obtained by substituting in (3.3.16), unbiased estimators of ω_1 and ω_2. Hence, the statistic $\mathcal{F}'$ can be used as an *approximate* test for testing the hypothesis of equal treatment effects. An exact test for testing the hypothesis $H_0 : \tau_1 = \cdots = \tau_v$ against the alternative that at least one pair of treatment effects is unequal has been developed by Cohen and Sackrowitz (1989). We refer to the original source for details on this test.

If the BIB design is resolvable, the block sum of squares in the intrablock analysis of variance table can be split into two components, viz., replication sum of squares and block within replication sum of squares. In such a case, an estimate of σ_b^2 can be obtained using the adjusted block within replications sum of squares. Let S_w^2 denote this sum of squares. Then, one can show that the expectation of $E_w = S_w^2/(b-r)$ is given by

$$\mathbb{E}(E_w) = \sigma^2 + (v-k)(r-1)(b-r)^{-1}\sigma_b^2. \tag{3.3.22}$$

Since $(b-r)/r = (v-k)/k$, we have

$$\mathbb{E}(E_w) = \sigma^2 + \frac{k(r-1)}{r}\sigma_b^2. \tag{3.3.23}$$

It follows then that

$$\mathbb{E}(rE_w - E_e) = (r-1)(\sigma^2 + k\sigma_b^2) \tag{3.3.24}$$

and thus,

$$\hat{\omega}_2 = \frac{r-1}{rE_w - E_e}. \tag{3.3.25}$$

Also, $\hat{\omega}_1 = E_e^{-1}$, as before.

As stated in Chapter 2, there are other approaches to the estimation of the weights ω_1 and ω_2 in the case of BIB designs as well. We refer to Graybill and Weeks (1959), Graybill and Deal (1959), Seshadri (1963), Shah (1964, 1970) for details on some of these.

3.4 Construction and Existence of BIB Designs

In this section, we describe some major methods of construction of BIB designs. The coverage however, is not intended to be encyclopedic. The two main methods of construction, viz., based on finite geometries and the method of differences, due to Bose (1939) are described in the following two subsections. Some other methods of construction are discussed in subsection 3.4.3. Finally, we present a few results on the existence of BIB designs in subsection 3.4.4.

3.4.1 BIB Designs Through Finite Geometries

In the Appendix, we have given a brief outline of finite projective and Euclidean geometries. These can be used to construct several families of BIB designs. To begin with, consider a finite projective plane of order $m = p^q$, where p is a prime and $q \geq 1$, an integer. There are m^2+m+1 points and the same number of lines in this geometry. Each line here contains $m+1$ distinct points, through each point there are exactly $m+1$ lines and each pair of points is joined by one line. Suppose we identify the points of the plane with treatments and the lines with blocks. Then we get the solution of a BIB design with parameters

$$v = m^2 + m + 1 = b, r = m + 1 = k, \lambda = 1. \tag{3.4.1}$$

More generally, consider a finite projective geometry $PG(n, m)$ where as before, $m = p^q$. To each point of $PG(n, m)$ let there correspond a treatment. Also, let a u-flat in the geometry be identified with a block. Following the notations in Section A.3 of the Appendix, the number of treatments and blocks are then given by

$$\begin{aligned} v &= \phi(n, 0, m) = \frac{m^{n+1} - 1}{m - 1} \\ b &= \phi(n, u, m) = \frac{(m^{n+1} - 1)(m^n - 1) \cdots (m^{n-u+1} - 1)}{(m^{u+1} - 1)(m^u - 1) \cdots (m - 1)}. \end{aligned} \tag{3.4.2}$$

The number of points in each u-flat is the number of (distinct) treatments contained in each block. Thus,

$$k = \phi(u, 0, m) = \frac{m^{u+1} - 1}{m - 1}. \tag{3.4.3}$$

The number of u-flats passing through a point equals the number of blocks containing a given treatment and therefore

$$r = \phi(n - 1, u - 1, m) = \frac{(m^n - 1)(m^{n-1} - 1) \cdots (m^{n-u+1} - 1)}{(m^u - 1)(m^{u-1} - 1) \cdots (m - 1)}. \tag{3.4.4}$$

Similarly, each pair of treatments is contained in λ blocks, where λ is the number of u-flats passing through a pair of points. Thus,

$$\lambda = \phi(n - 2, u - 2, m) = \frac{(m^{n-1} - 1)(m^{n-2} - 1) \cdots (m^{n-u+1} - 1)}{(m^{u-1} - 1)(m^{u-2} - 1) \cdots (m - 1)}. \tag{3.4.5}$$

We therefore have the following result.

Theorem 3.4.1 *Identifying the points of a finite projective geometry $PG(n, m)$ with treatments and its u-flats as blocks, one gets a BIB design with parameters given by (3.4.2)–(3.4.5).*

The family of BIB designs with parameters as in (3.4.1) is a special case of the family given in Theorem 3.4.1 with $n = 2, u = 1$. For

$n = 3, u = 2$ in Theorem 3.4.1, we get a family of BIB designs with parameters

$$v = m^3 + m^2 + m + 1 = b, r = m^2 + m + 1 = k, \lambda = m + 1. \quad (3.4.6)$$

Similarly, with $n = 3, u = 1$, we get a family of BIB designs with parameters

$$\begin{aligned} v &= (m+1)(m^2+1), b = (m^2+1)(m^2+m+1), r = m^2+m+1, \\ k &= m+1, \lambda = 1. \end{aligned} \quad (3.4.7)$$

Example 3.4.1 Let $m = 2$ in the family with parameters (3.4.6). For this value of m the parameters are $v = 15 = b, r = 7 = k, \lambda = 3$. In $PG(3,2)$, each point is represented by a 4-tuple (y_0, y_1, y_2, y_3), $y_i = 0$ or 1 for all $i, (y_0, y_1, y_2, y_3) \neq (0,0,0,0)$. Since $u = 2$ here, we consider the 2-flats in $PG(3,2)$, whose equations are as under: $y_i = 0,\ 0 \leq i \leq 3$; $y_i + y_j = 0,\ i \neq j, i, j = 0,1,2,3$; $y_i + y_j + y_k = 0, i \neq j \neq k, i, j, k = 0,1,2,3$; $y_0 + y_1 + y_2 + y_3 = 0$, all additions being modulo 2.

A solution of the required BIB design is obtained by writing down the points lying on these 2-flats. The blocks are shown below where a treatment is represented by a 4-tuple.

(0001	0010	0011	0100	0101	0110	0111)
(0001	0010	0011	1000	1001	1010	1011)
(0001	0101	0100	1000	1001	1100	1101)
(0010	0100	0110	1000	1010	1100	1110)
(0001	0010	0011	1100	1101	1110	1111)
(0001	0100	0101	1010	1011	1110	1111)
(0010	0100	0110	1001	1011	1101	1111)
(0001	1000	1001	0110	0111	1110	1111)
(0010	1000	1010	0101	0111	1101	1111)
(0100	1000	1100	0011	0111	1011	1111)
(0001	0110	0111	1010	1011	1100	1101)
(0101	1001	1100	0010	0111	1011	1110)
(0011	1001	1010	0100	0111	1101	1110)
(0011	0101	0110	1000	1011	1101	1110)
(0110	0101	1001	1010	1100	0011	1111)

We can associate the treatment $8y_0 + 4y_1 + 2y_2 + y_3$ to the point (y_0, y_1, y_2, y_3) to get the treatments labels as $1, 2, \ldots, 15$.

Consider now a finite Euclidean geometry $EG(n,m)$, where m is a prime or a prime power. To every point of $EG(n,m)$ let there correspond a treatment and as in the case of finite projective geometry, let a u-flat correspond to a block. With this correspondence, the number of treatments and blocks are

$$v = m^n, \tag{3.4.8}$$

$$\begin{aligned} b &= m^{n-u}\phi(n-1,u-1,m) \\ &= \frac{m^{n-u}(m^n-1)(m^{n-1}-1)\cdots(m^{n-u+1}-1)}{(m^u-1)(m^{u-1}-1)\cdots(m-1)}. \end{aligned} \tag{3.4.9}$$

The number of points lying on a u-flat is the block size and thus

$$k = m^u. \tag{3.4.10}$$

The number of u-flats through a point is the replication of a treatment and hence

$$\begin{aligned} r &= \phi(n-1,u-1,m) \\ &= \frac{(m^n-1)(m^{n-1}-1)\cdots(m^{n-u+1}-1)}{(m^u-1)(m^{u-1}-1)\cdots(m-1)}. \end{aligned} \tag{3.4.11}$$

On similar lines, one can show that

$$\begin{aligned} \lambda &= \phi(n-2,u-2,m) \\ &= \frac{(m^{n-1}-1)(m^{n-2}-1)\cdots(m^{n-u+1}-1)}{(m^{u-1}-1)(m^{u-2}-1)\cdots(m-1)}. \end{aligned} \tag{3.4.12}$$

Summarizing, we then have the following result.

Theorem 3.4.2 *Identifying the points of a finite Euclidean geometry $EG(n,m)$ with treatments and its u-flats as blocks, one gets a BIB design with parameters given by (3.4.8)–(3.4.12).*

Example 3.4.2 To illustrate the above theorem, let us take $m = 2, n = 3, u = 2$. From (3.4.8)–(3.4.12), the parameters of the BIB design are then $v = 8, b = 14, r = 7, k = 4, \lambda = 3$. Identifying the points of an $EG(3,2)$ with the treatments and the 2-flats (or, planes) with the blocks, we get a solution of the required BIB design. The full design is shown below, where the point (y_1, y_2, y_3) of $EG(3,2)$ is identified with the treatment label $4y_1 + 2y_2 + y_3$.

$$\begin{array}{ll}
(0,2,1,3); & (4,6,5,7); \\
(0,4,1,5); & (2,6,3,7); \\
(0,4,2,6); & (1,5,3,7); \\
(0,3,4,7); & (2,1,6,5); \\
(0,5,2,7); & (4,1,6,3); \\
(0,6,1,7); & (4,2,5,3); \\
(0,3,5,6); & (4,2,1,7).
\end{array}$$

Note that this design is resolvable, the two blocks in each row forming a complete replication. In fact, the solution given above is affine resolvable.

The design in the above example is a member of the family

$$v = m^3, b = m(m^2+m+1), r = m^2+m+1, k = m^2, \lambda = m+1 \quad (3.4.13)$$

obtained by taking $n = 3$ and $u = 2$ in Theorem 3.4.2. Similarly, taking $n = 2, u = 1$ in Theorem 3.4.2, one gets a BIB design with parameters

$$v = m^2, b = m^2 + m, r = m + 1, k = m, \lambda = 1, \quad (3.4.14)$$

which is also called a (finite) affine plane.

Remark 3.4.1 An alternative method of obtaining the BIB designs with parameters (3.4.14) utilizes a complete set of *mutually orthogonal Latin squares* (MOLS). Recall that a Latin square of order $m(\geq 2)$ is an $m \times m$ array, with entries from a set of m distinct symbols such that each symbol appears exactly once in each row and once in each column of the array. Two Latin squares of the same order are said to be orthogonal to each other if, when any one of the squares is superimposed on the other, every ordered pair of symbols appears exactly once. A set of Latin squares is said to form a set of MOLS if every pair in the set is orthogonal to each other. The maximum number of MOLS of order $m(> 2)$ is $(m - 1)$, this number being attainable if m is a prime or a prime power (see e.g., Raghavarao (1971, Chapter 1)) and in such a case we say that there is a *complete set* of MOLS.

In order to obtain a solution of the family of BIB designs with parameters as in (3.4.14), assume that $m \geq 2$ is a prime or a prime power and thus, the existence of a complete set of MOLS is guaranteed; let these Latin squares be denoted by $L = \{L_1, \ldots, L_{m-1}\}$. First, arrange the $v = m^2$ treatments in an $m \times m$ array, say L_0 in any order. Obtain m blocks, each of size $k = m$ by treating the rows of L_0 as blocks. Another m blocks of size m each are obtained by treating the columns of L_0 as blocks. Next, superimpose one of the Latin squares, say L_i, from the set L on L_0 and form a block by including all those treatments which fall under a particular letter of L_i. Since there are m letters in L_i, there will be m blocks when L_i is superimposed on L_0 and blocks formed as indicated above. Repeat the same procedure with each of the Latin squares in L. This procedure generates a totality of $b = m + m + m(m-1) = m^2 + m$ blocks and these blocks provide a solution of the BIB designs with parameters given by (3.4.14). The family of BIB designs with parameters in (3.4.14) is also known as "Yates' orthogonal series".

3.4.2 Method of Differences

The method of symmetrically repeated differences is a very powerful method of construction of incomplete block designs and, in particular of BIB designs. Consider a finite additive Abelian group $\mathcal{M}$ with n elements. To each element of $\mathcal{M}$, let there correspond m treatments, the treatments corresponding to an element $a \in \mathcal{M}$ being denoted by $a_1, \ldots, a_m$. The treatment a_i is said to belong to the ith class. Clearly, we have mn treatments in all, n of these belonging to each of the m classes. With any ordered pair of distinct treatments a_i and b_j, we associate the *difference* $a - b$ of the type $[i, j]$. Each difference is an element of $\mathcal{M}$ and is of a certain type. If $i = j$, the difference is called a *pure* difference. Obviously, in such a case, we must have $a \neq b$, as the treatments are distinct. Similarly, if $i \neq j$, the difference is said to be a *mixed* difference. For example, consider the additive Abelian group consisting of residue classes mod 5. To each element of this group, let there correspond two treatments, a_1 and a_2. Then, the difference associated with the ordered pair of treatments 2_1 and 3_1 is the pure difference 4 of the type $[1, 1]$ as, $2 - 3 = -1 = 4 \pmod 5$ whereas the difference associated with the pair 2_2 and 4_1 is the mixed difference 3 of type $[2, 1]$.

Suppose now that there is a block B containing k distinct treatments. From this block, one can get $k(k-1)$ ordered pairs of treatments, giving rise to $k(k-1)$ differences. These differences are said to be arising out of B. Continuing with the example in the previous paragraph, suppose $B = (2_1, 4_2, 0_2)$. The differences arising out of B are the pure differences 4 and 1 of the type $[2, 2]$ and the mixed differences 3 and 2 of the type

[1, 2] and mixed differences 2 and 3 of the type [2, 1].

Since there are m classes and $\mathcal{M}$ has n elements, there are $(n-1)$ pure differences of the type $[i,i]$ for $1 \le i \le m$ and there are n mixed differences of each type $[i,j]$, $i \ne j$, $1 \le i,j \le m$. Suppose now that we have a set of t blocks $B_1, \ldots, B_t$, each containing k distinct treatments. If among the differences arising out of these t blocks, each possible difference appears λ times, then the differences are said to be *symmetrically repeated.* As an example, again consider $\mathcal{M}$ to be the Abelian group of residue classes mod 5 and to each element of $\mathcal{M}$ let there correspond three treatments, a_1, a_2, a_3, $a \in \mathcal{M}$. Next, consider the following six blocks:

$$(0_1, 1_1, 0_2); (0_2, 1_2, 2_3); (0_3, 1_3, 2_1); (0_1, 2_1, 3_2); (0_2, 2_2, 0_3); (0_3, 2_3, 0_1). \tag{3.4.15}$$

The differences arising out of these 6 blocks can be displayed as in Table 3.4.1.

TABLE 3.4.1: Differences From the Blocks (3.4.15)

Blocks	Differences of type								
	[1,1]	[2,2]	[3,3]	[1,2]	[1,3]	[2,3]	[2,1]	[3,1]	[3,2]
$(0_1, 1_1, 0_2)$	4,1	-	-	0,1	-	-	0,4	-	-
$(0_2, 1_2, 2_3)$	-	4,1	-	-	-	3,4	-	-	2,1
$(0_3, 1_3, 2_1)$	-	-	4,1	-	2,1	-	-	3,4	-
$(0_1, 2_1, 3_2)$	3,2	-	-	2,4	-	-	3,1	-	-
$(0_2, 2_2, 0_3)$	-	3,2	-	-	-	0,2	-	-	0,3
$(0_3, 2_3, 0_1)$	-	-	3,2	-	0,3	-	-	0,2	-

From Table 3.4.1, we see that among the differences arising out of the blocks given in (3.4.15), the differences are *not* repeated symmetrically. However, if we add a seventh block $(0_1, 2_2, 1_3)$ to the above 6 blocks, it seen that the differences are symmetrically repeated, each appearing exactly once.

Consider again a finite additive Abelian group $\mathcal{M}$ having n elements, say $y^{(0)}, y^{(1)}, \ldots, y^{(n-1)}$, and to each element $y^{(u)}$ let there correspond m treatments $y_1^{(u)}, y_2^{(u)}, \ldots, y_m^{(u)}$, $0 \le u \le n-1$. As before, the treatment $y_i^{(u)}$ is said to belong to the ith class. Suppose B_α is a given block containing k of these treatments, the treatments in the given block being all distinct. From B_α one can get n blocks $B_{\alpha,\,\delta}$, where δ ranges over the elements of $\mathcal{M}$, as follows: Corresponding to any treatment $y_i^{(u)}$ of the ith class appearing in B_α, we take the treatment $y_i^{(v)}$ of the ith class in $B_{\alpha,\,\delta}$, where $y^{(v)} = y^{(u)} + \delta$. The n blocks $\{B_{\alpha,\,\delta}\}$ are said to be obtained by *developing* the block B_α. Clearly, $B_{\alpha,\,0} = B_\alpha$. We are now in a position to state the following result, called the first fundamental theorem, due to Bose (1939).

Theorem 3.4.3 *Consider the set of treatments* $y_1^{(u)}, y_2^{(u)}, \ldots, y_m^{(u)}$, $0 \leq u \leq n-1$. *Suppose it is possible to find a set of* t *blocks* $B_1, \ldots, B_t$ *satisfying the following conditions:*
(a) Each block contains k *distinct treatments,*
(b) among the kt *treatments occurring in the* t *blocks, exactly* r *treatments belong to each of the classes, and*
(c) the differences arising from the t *blocks are symmetrically repeated,* λ *times each.*

Then, the nt *blocks obtained by developing the initial* t *blocks* $B_1, \ldots, B_t$ *provide a solution of a BIB design with parameters* $v = mn$, $b = nt, r, k, \lambda$.

Proof. Corresponding to a treatment $y_i^{(u)}$ of the ith class in B_α, $1 \leq \alpha \leq t$, we have a treatment $y_i^{(v)}$ in $B_{\alpha,\delta}$ where $y^{(v)} = y^{(u)} + \delta$. As δ ranges over all the elements of $\mathcal{M}$, $y^{(v)}$ also ranges over all the elements of $\mathcal{M}$. Hence, corresponding to the treatment $y_i^{(u)}$ in B_α, we have each treatment of the ith class occurring precisely once in the blocks $B_{\alpha,\delta}$. From condition (b) of the theorem, it follows that each treatment occurs precisely r times in the design. A pair of treatments $y_i^{(v)}$ and $y_i^{(v')}$ belonging to the same class may be termed as pure pair while treatments $y_i^{(v)}$ and $y_j^{(v')}$ belonging to two different classes may be called a mixed pair. The treatments $y_i^{(v)}$ and $y_i^{(v')}$ occur together in some block of the design if and only if we are able to find a pair of treatments $y_i^{(u)}$ and $y_i^{(u')}$ occurring together in some initial block B_s, $1 \leq s \leq t$ and an element $\delta \in \mathcal{M}$ such that

$$y^{(v)} = y^{(u)} + \delta, \quad y^{(v')} = y^{(u')} + \delta. \tag{3.4.16}$$

Then,

$$y^{(u)} - y^{(u')} = y^{(v)} - y^{(v')} = \text{a fixed element of } \mathcal{M}. \tag{3.4.17}$$

From condition (c), it follows that it is possible to find treatments $y_i^{(u)}$ and $y_i^{(u')}$ belonging to some block B_s in exactly λ ways so that (3.4.17) is satisfied. This shows that every pure pair of treatments occurs together in λ blocks. Using a similar argument, the same can be shown about each mixed pair. □

We now illustrate the above theorem.

Example 3.4.3 Consider the 6 initial blocks given by (3.4.15) along with the additional block $(0_1, 2_2, 1_3)$. It has already been observed that these 7 blocks give rise to symmetrically repeated differences with $\lambda = 1$. Also, one can verify that conditions (a) and (b) of Theorem 3.4.3 hold. Thus, these 7 initial blocks when developed lead to a solution of the BIB design with parameters $v = 15, b = 35, r = 7, k = 3, \lambda = 1$. The full design is shown below.

$$\begin{array}{lllll}
(0_1, 1_1, 0_2), & (1_1, 2_1, 1_2), & (2_1, 3_1, 2_2), & (3_1, 4_1, 3_2), & (4_1, 0_1, 4_2), \\
(0_2, 1_2, 2_3), & (1_2, 2_2, 3_3), & (2_2, 3_2, 4_3), & (3_2, 4_2, 0_3), & (4_2, 0_2, 1_3), \\
(0_3, 1_3, 2_1), & (1_3, 2_3, 3_1), & (2_3, 3_3, 4_1), & (3_3, 4_3, 0_1), & (4_3, 0_3, 1_1), \\
(0_1, 2_1, 3_2), & (1_1, 3_1, 4_2), & (2_1, 4_1, 0_2), & (3_1, 0_1, 1_2), & (4_1, 1_1, 2_2), \\
(0_2, 2_2, 0_3), & (1_2, 3_2, 1_3), & (2_2, 4_2, 2_3), & (3_2, 0_2, 3_3), & (4_2, 1_2, 4_3), \\
(0_3, 2_3, 0_1), & (1_3, 3_3, 1_1), & (2_3, 4_3, 2_1), & (3_3, 0_3, 3_1), & (4_3, 1_3, 4_1), \\
(0_1, 2_2, 1_3), & (1_1, 3_2, 2_3), & (2_1, 4_2, 3_3), & (3_1, 0_2, 4_3), & (4_1, 1_2, 0_3).
\end{array}$$

Example 3.4.4 Let $\mathcal{M}$ be the additive Abelian group of residue classes modulo 13 and, to each element of $\mathcal{M}$, let there correspond just one treatment. The treatments are thus labeled as $0, 1, \ldots, 12$. Consider the initial block $(0, 1, 3, 9)$. It is easy to see that among the differences arising from this block, every nonzero element of $\mathcal{M}$ appears exactly once. Therefore, by developing this block we get a BIB design with parameters $v = 13 = b, r = 4 = k, \lambda = 1$.

For presenting the next main result on the method of differences, we need some additional notation and terminology. As before, let $\mathcal{M}$ denote a finite additive Abelian group containing n elements $y^{(0)}, \ldots, y^{(n-1)}$ and to an element $y^{(u)}$ let there correspond m treatments $y_1^{(u)}, \ldots, y_m^{(u)}$, $0 \leq u \leq n-1$. To these mn treatments, we adjoin another treatment ∞, called an invariant treatment, so that we now have $v = mn + 1$ treatments. Given any block B_α containing k distinct treatments, we can obtain n blocks $B_{\alpha,\delta}$, $\delta \in \mathcal{M}$, from it as explained earlier. If ∞ appears in B_α then it also appears in $B_{\alpha,\delta}$. The n blocks $B_{\alpha,\delta}$ are said to be obtained by developing B_α. The next result, called the second fundamental theorem due to Bose (1939), can be proved on the lines of Theorem 3.4.3.

Theorem 3.4.4 *Consider the set of treatments $y_1^{(u)}, y_2^{(u)}, \ldots, y_m^{(u)}$, $0 \leq u \leq n-1$ together with the invariant treatment ∞. Suppose it is possible to find a set of $t+s$ blocks $B_1, \ldots, B_t, B_1', \ldots, B_s'$ satisfying the following conditions:*

(a) Each block B_i, $1 \le i \le t$, contains k distinct treatments and each block $B'_j, 1 \le j \le s$, contains ∞ and $k-1$ distinct treatments $y_i^{(u)}$,
(b) among the kt treatments occurring in the t blocks $\{B_i\}$, exactly $ns-\lambda$ belong to each of the m classes, whereas among the $s(k-1)$ treatments occurring in the blocks $\{B'_j\}$, exactly λ belong to each of the m classes, and
(c) the differences arising from the $t+s$ blocks $B_1, \ldots, B_t, B_1^*, \ldots, B_s^*$ are symmetrically repeated, λ times each, where for $1 \le j \le s$, the block B_j^* is obtained from B'_j by deleting ∞.

Then, the $n(t+s)$ blocks obtained by developing the initial $t+s$ blocks provide a solution of a BIB design with parameters $v = mn+1, b = n(t+s), r = ns, k, \lambda$.

Based on Theorems 3.4.3 and 3.4.4, we now proceed to obtain some specific families of BIB designs.

(i) *BIB designs with $k = 3, \lambda = 1$.*

Such BIB designs are known as Steiner's triple systems. For such BIB designs, we have from (3.2.1)

$$3b = vr \text{ and } v - 1 = 2r. \tag{3.4.18}$$

It follows then r must be either of the form $3m+1$ or $3m$, for some integer m. In fact, a Steiner's triple system on v treatments exists *if and only if* $v \equiv 1, 3 \pmod 6$ (Kirkman (1847)). We thus get the following two families of Steiner's triple systems:

Family I : $v = 6m+3, b = (3m+1)(2m+1), r = 3m+1, k = 3, \lambda = 1$.
Family II : $v = 6m+1, b = m(6m+1), r = 3m, k = 3, \lambda = 1$.

For constructing the designs in Family I, consider the following sets of initial blocks:

$$\begin{gathered}[1_1, (2m)_1, 0_2], [2_1, (2m-1)_1, 0_2], \cdots, [m_1, (m+1)_1, 0_2], \\ [1_2, (2m)_2, 0_2], [2_2, (2m-1)_2, 0_3], \cdots, [m_2, (m+1)_2, 0_3], \\ [1_3, (2m)_3, 0_1], [2_3, (2m-1)_3, 0_1], \cdots, [m_3, (m+1)_3, 0_1], \\ [0_1, 0_2, 0_3].\end{gathered} \tag{3.4.19}$$

Then one can prove the following result using Theorem 3.4.3.

Theorem 3.4.5 *The initial blocks given in (3.4.19) provide a solution of the BIB designs belonging to Family* I *with parameters $v = 6m+3, b = (3m+1)(2m+1), r = 3m+1, k = 3, \lambda = 1$.*

Example 3.4.5 Let $m = 1$ in Family I. Then the design parameters are $v = 9, b = 12, r = 4, k = 3, \lambda = 1$. The initial blocks are

$$(1_1, 2_1, 0_2), (1_2, 2_2, 0_3), (1_3, 2_3, 0_1), (0_1, 0_2, 0_3).$$

The full design is obtained by developing these initial blocks and is shown below:

$$\begin{array}{lll} (1_1, 2_1, 0_2) & (2_1, 0_1, 1_2) & (0_1, 1_1, 2_2) \\ (1_2, 2_2, 0_3) & (2_2, 0_2, 1_3) & (0_2, 1_2, 2_3) \\ (1_3, 2_3, 0_1) & (2_3, 0_3, 1_1) & (0_3, 1_3, 2_1) \\ (0_1, 0_2, 0_3) & (1_1, 1_2, 1_3) & (2_1, 2_2, 2_3). \end{array}$$

We next present a method of construction of designs belonging to Family II when $v = 6m + 1$ is a prime or a prime power. Let $GF(v)$ denote the Galois field of order $v = 6m+1$ and x be a primitive element of $GF(v)$. Consider the m initial blocks

$$(x^i, x^{2m+i}, x^{4m+i}), \; 0 \le i \le m - 1. \tag{3.4.20}$$

It can then be shown that among the differences arising out of the blocks in (3.4.20), each nonzero element of $GF(v)$ appears precisely once. We thus have the following result.

Theorem 3.4.6 *Let $v = 6m+1$ be a prime or a prime power. Then the initial blocks in (3.4.20) provide a solution to the BIB designs of Family II.*

Example 3.4.6 Let $m = 3$ so that there are $6m + 1 = 19$ treatments. Since $x = 2$ is a primitive element of $GF(19)$, the initial blocks are

$$\begin{aligned} (2^0, 2^6, 2^{12}) &= (1, 7, 11) \\ (2^1, 2^7, 2^{13}) &= (2, 14, 3) \\ (2^2, 2^8, 2^{14}) &= (4, 9, 6). \end{aligned}$$

Developing these three initial blocks, we get the solution of a BIB design with parameter $v = 19, b = 57, r = 9, k = 3, \lambda = 1$.

For a general method of construction of Steiner's triple systems with $v \equiv 1 \pmod 6$, see Skolem (1958). A more recent review on Steiner's triples is by Colbourn (2007).

(ii) *A Family of Symmetric BIB Designs.*

Consider a family of symmetric BIB designs satisfying $r = (v-1)/2$. Since the design is symmetric, we have

$$\lambda(v-1) = r(r-1) = \frac{(v-1)(v-3)}{4}.$$

It follows then that $v = 4\lambda + 3$. Let $\lambda = u - 1$ for some integer $u > 1$. The parameters of the symmetric BIB design are then

$$v = 4u - 1 = b, r = 2u - 1 = k, \lambda = u - 1. \tag{3.4.21}$$

We first obtain a solution of the BIB designs with parameters in (3.4.21) when $v = 4u-1$ is a prime or a prime power. Let x be a primitive element of $GF(4u-1)$. The nonzero elements of $GF(4u-1)$ can be written as $x^0 = 1, x, x^2, \ldots, x^{4u-3}$. Consider the initial block containing all the even powers of x, viz.,

$$(x^0, x^2, x^4, \ldots, x^{4u-4}). \tag{3.4.22}$$

It can be seen that among the differences arising from the block (3.4.22), each nonzero element of $GF(4u-1)$ appears exactly $u-1$ times. Hence, we have the following result.

Theorem 3.4.7 *If $v = 4u - 1$ is a prime or a prime power, the initial block (3.4.22) provides a solution of the BIB designs with parameters in (3.4.21).*

Example 3.4.7 Let $u = 2$ in (3.4.21). Since $x = 3$ is a primitive element of $GF(7)$, from Theorem 3.4.7, the initial block in this case is $(3^0, 3^2, 3^4) = (1, 2, 4)$. By developing this initial block, we get a solution of a BIB design with parameters $v = 7 = b, r = 3 = k, \lambda = 1$. The full design is already exhibited in Example 3.2.1.

Alternatively, a solution of the BIB designs with parameters in (3.4.21) can be obtained by developing the initial block containing all odd powers of the primitive element x, viz., $(x, x^3, \ldots, x^{4u-3})$.

A solution of the family of designs with parameters in (3.4.21) can sometimes be obtained even when $v = 4u - 1$ is not a prime or a prime power. This solution is dependent on the existence of an *Hadamard matrix* of order $4u$.

Definition 3.4.1 *A square matrix H_n of order n with entries ± 1 is called an Hadamard matrix if $H_n H_n' = nI_n$.*

If H_n is an Hadamard matrix, then it is easily seen that $H_n' H_n = nI_n$. It follows then that an Hadamard matrix remains so when any row or column is multiplied by -1. In view of this, one can always write an Hadamard matrix with its first row and first column containing only $+1$'s and then we say that the Hadamard matrix is in its normal form.

Trivially, H_n exists for $n = 1$ and H_2 is given by

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

It can be shown that a necessary condition for the existence of an Hadamard matrix H_n, $n > 2$ is that $n \equiv 0 \pmod 4$; for a proof of this fact, see e.g., Hall (1986). It is not known as yet whether this necessary condition is sufficient as well. Hadamard matrices for all permissible values of $n \leq 100$, with the exception of $n = 92$ are displayed in Plackett and Burman (1946). An Hadamard matrix of order 92 was discovered by Baumert, Golomb and Hall (1962). Hadamard matrices for all permissible values of $n \leq 424$ are now known to exist. If H_m and H_n are Hadamard matrices of orders m and n respectively, then their tensor product $H_m \otimes H_n$ is an Hadamard matrix of order mn. In particular, an Hadamard matrix H_n of order n where $n = 2^s$ and $s \geq 2$ is an integer, can be constructed by taking the s-fold tensor product of H_2 with itself, i.e.,

$$H_{2^s} = \underbrace{H_2 \otimes H_2 \otimes \cdots \otimes H_2}_{s \text{ times}}.$$

Consider now an Hadamard matrix H_{4u}, which without loss of generality, is assumed to be in its normal form. Delete from H_{4u} its first row and first column of all ones to obtain a matrix A of order $(4u-1) \times (4u-1)$. Define

$$N = \frac{1}{2}(A + J_{4u-1}). \tag{3.4.23}$$

This means that N is obtained from A by replacing the -1's in A by zero and keeping $+1$'s unaltered. Then, it is not hard to see that N is the incidence matrix of a BIB design with parameters as in (3.4.21). Conversely, if M is the incidence matrix of a BIB design with parameters given by (3.4.21), then by replacing the zeros in M by -1 and bordering the resultant matrix by a row and column of all ones, one gets an Hadamard matrix of order $4u$. We thus have the following

Theorem 3.4.8 *The existence of an Hadamard matrix of order* $4u$ *is equivalent to the existence of a BIB design with parameters given by (3.4.21).*

Example 3.4.8 Consider an Hadamard matrix H_{16} which can be obtained by forming the tensor product $H_4 \otimes H_4$, where H_4 is as below:

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Following the construction method described above, we get a solution of a BIB design with parameters $v = 15 = b, r = 7 = k, \lambda = 3$. Details are left for the reader.

(iii) *Some Other Families of BIB Designs*

Let $v = 4s + 1$ be a prime or a prime power and let x be a primitive element of $GF(4s + 1)$. Consider the following two initial blocks:

$$(x^0, x^2, x^4, \ldots, x^{4s-2}) \text{ and } (x^1, x^3, x^5, \ldots, x^{4s-1}). \tag{3.4.24}$$

One can see that these initial blocks on development provide a solution of a BIB design with parameters

$$v = 4s + 1, b = 8s + 2, r = 4s, k = 2s, \lambda = 2s - 1. \tag{3.4.25}$$

Example 3.4.9 Let $s = 4$ in (3.4.25), so that $v = 17$. A primitive element of $GF(17)$ is $x = 3$ (see Section A.3). The initial blocks, given by (3.4.24) are therefore

$$\begin{aligned} (3^0, 3^2, 3^4, 3^6, 3^8, 3^{10}, 3^{12}, 3^{14}) &= (1, 9, 13, 15, 16, 8, 4, 2) \\ (3^1, 3^3, 3^5, 3^7, 3^9, 3^{11}, 3^{13}, 3^{15}) &= (3, 10, 5, 11, 14, 7, 12, 6). \end{aligned}$$

By developing the above two blocks, we get a solution of a BIB design with parameters $v = 17, b = 34, r = 16, k = 8, \lambda = 7$.

Next, let $v = m(u - 1) + 1$ be a prime or a prime power, where m, u are integers and suppose x is a primitive element of $GF(v)$. Then, as shown by Sprott (1954), the m initial blocks

$$(0, x^i, x^{m+i}, x^{2m+i}, \ldots, x^{m(u-2)+i}),\ 0 \le i \le m - 1 \tag{3.4.26}$$

when developed, lead to a solution of a BIB design with parameters

$$v = m(u-1)+1, b = m^2(u-1)+m, r = mu, k = u = \lambda. \quad (3.4.27)$$

For several other methods of construction of BIB designs through the method of differences, we refer to Hall (1986) and Raghavarao (1971). A table of difference set solutions of BIB designs was provided by Takeuchi (1962). See also Kageyama (1972) in this context.

3.4.3 Some Other Constructions

It was shown in Section 3.3 that given a symmetric BIB design one can obtain two more BIB designs from the parent one by the processes of block section and block intersection. Thus, for example, starting with the BIB design with parameters $v = 4u-1 = b, r = 2u-1 = k, \lambda = u-1$, one can get the solutions of BIB designs with parameters

$$v = 2u, b = 4u-2, r = 2u-1, k = u, \lambda = u-1 \quad (3.4.28)$$

and

$$v = 2u-1, b = 4u-2, r = 2u-2, k = u-1, \lambda = u-2. \quad (3.4.29)$$

A special family of BIB designs, called Family(A) BIB designs with parameters v, b, r, k, λ are characterized by the condition

$$b = 4(r-\lambda). \quad (3.4.30)$$

Shrikhande (1962) proved that designs belonging to Family(A) reproduce themselves under a certain type of composition. In this context, we have the following result.

Theorem 3.4.9 *For $i = 1, 2$, let N_i be the incidence matrix of a BIB design d_i belonging to Family(A) with parameters $v_i, b_i, r_i, k_i, \lambda_i$ and let $\bar{N}_i = J_{v_i b_i} - N_i$ be the incidence matrix of $\bar{d}_i$, the complement of d_i. Then,*

$$N_d = N_1 \otimes N_2 + \bar{N}_1 \otimes \bar{N}_2 \quad (3.4.31)$$

is the incidence matrix of a BIB design d belonging to Family(A). The parameters of d are $v = v_1 v_2, b = b_1 b_2, r = r_1 r_2 + (b_1 - r_1)(b_2 - r_2), k = k_1 k_2 + (v_1 - k_1)(v_2 - k_2), \lambda = r - b/4$.

Proof. Let v denote the number of treatments and b, the number of blocks in d. First observe that the matrix N_d in (3.4.31) is obtained by replacing in N_1, the ones by N_2 and the zeros by $\bar{N}_2$. Clearly then, N_d is a $v \times b$ matrix with entries 0 and 1 where

$$v = v_1 v_2 \text{ and } b = b_1 b_2. \tag{3.4.32}$$

It is easy to see that each row sum of N_d is

$$r = r_1 r_2 + (b_1 - r_1)(b_2 - r_2) \tag{3.4.33}$$

and each column sum of N_d is

$$k = k_1 k_2 + (v_1 - k_1)(v_2 - k_2). \tag{3.4.34}$$

Let us now label the rows of N_d as (s, t), $1 \le s \le v_1, 1 \le t \le v_2$. It is not hard to see that the inner product of two rows of N_d with labels (s, t) and (s, t') where $t \neq t'$ is

$$r_1 \lambda_2 + (b_1 - r_1)(b_2 - 2r_2 + \lambda_2) = \alpha_1, \text{ say.} \tag{3.4.35}$$

Similarly, it can be seen that the inner product of two rows of N_d with labels (s, t) and (s', t) where $s \neq s'$ is

$$(b_1 - 2r_1 + \lambda_1)(b_2 - r_2) + \lambda_1 r_2 = \alpha_2. \tag{3.4.36}$$

Finally, consider two rows of N_d with labels (s, t) and (s', t') where $s \neq s', t \neq t'$. The inner product of these two rows is seen to be

$$\lambda_1\lambda_2 + 2(r_1 - \lambda_1)(r_2 - \lambda_2) + (b_1 - 2r_1 + \lambda_1)(b_2 - 2r_2 + \lambda_2) = \alpha_3. \tag{3.4.37}$$

Since for $i = 1, 2$, d_i belongs to Family(A), we have $b_i = 4(r_i - \lambda_i)$. Using these conditions and (3.4.35)–(3.4.37), we can see that $\alpha_1 = \alpha_2 = \alpha_3 = \lambda = r - b/4$. □

We have described in this section some important methods of construction of BIB designs. However, there are several other methods of construction of BIB designs on which we do not elaborate here. For more details on the construction of BIB designs, one might refer to Hall (1986), Raghavarao (1971), Street and Street (1987) and Beth, Jungnickel and Lenz (1993).

It might be noted that given positive integers v and k, $2 \le k < v$, a BIB design with v treatments in blocks of size k can always be constructed. Form blocks by taking k elements out of the set of v treatments in all possible ways. These blocks constitute a BIB design with parameters $v, b = \binom{v}{k}, r = \binom{v-1}{k-1}, k, \lambda = \binom{v-2}{k-2}$. Such BIB designs are called *unreduced.* A table of BIB designs with $k \le v/2$ and $3 \le r \le 15$ is available in Beth *et al.* (1993).

3.4.4 Existence of BIB Designs

In closing this section, we take up the issue of existence of BIB designs. Recall that the relations among the parameters of BIB designs, viz., $vr = bk, r(k-1) = \lambda(v-1)$ and $b \geq v$ are merely necessary for the existence of a BIB design and are *not* sufficient in general; these are sufficient for $k = 2, 3$. General sufficient conditions for the existence of BIB designs are yet to be found. In specific cases, additional necessary conditions have been found and we describe some of these results now. Throughout we take $k \geq 2$. The following result was first proved in the special case $\lambda = 1$ by Bruck and Ryser (1949). The result in its full generality was proved by Chowla and Ryser (1950) and independently by Shrikhande (1950).

Theorem 3.4.10 *For a symmetric BIB design with parameters $v = b, r = k, \lambda$, and v even, $r - \lambda$ must be a perfect square.*

Proof. Let d be a symmetric BIB design with parameters $v = b, r = k, \lambda$ where v is *even*. Let N_d be the incidence matrix of d. Then, we have

$$\det(N_d N_d') = rk(r-\lambda)^{v-1} = r^2(r-\lambda)^{v-1}.$$

Note that N_d is a square matrix of order v as d is symmetric and thus,

$$\det(N_d N_d') = \{\det(N_d)\}^2 = r^2(r-\lambda)^{v-1} \Rightarrow \det(N_d) = \pm r(r-\lambda)^{\frac{v-1}{2}}.$$

Since N_d is a matrix with integral entries, its determinant must be an integer, which is possible only if $(r - \lambda)$ is a perfect square. □

In view of Theorem 3.4.10, the following symmetric BIB designs with $k \leq 15$ cannot be constructed or, in other words, are *non-existent.*
$v = 22, k = 7, \lambda = 2$; $v = 46, k = 10, \lambda = 2$; $v = 34, k = 12, \lambda = 4$; $v = 92, k = 14, \lambda = 2$; $v = 106, k = 15, \lambda = 2$.

The next result, which we state without proof, provides an additional necessary condition on the existence of a symmetric BIB design when the number of treatments v is *odd.* This result is known as the Bruck-Ryser-Chowla theorem.

Theorem 3.4.11 *A necessary condition for the existence of a symmetric BIB design with parameters $v = b, r = k, \lambda$ where v is odd is that the equation*

$$x^2 = (r-\lambda)y^2 + (-1)^{\frac{v-1}{2}}\lambda z^2$$

has a solution in integers x, y, z, not all simultaneously zero.

As an application of the above theorem, let $v = 43, k = 7, \lambda = 1$. From Theorem 3.4.11, a necessary condition for the existence of this design is that $x^2 = 6y^2 - z^2$ has a solution in integers x, y, z. However, it is not hard to see that no such solution exists and thus the symmetric BIB design with parameters $v = 43, k = 7, \lambda = 1$ does not exist.

We have already seen in this section that a BIB design with parameters $v = m^2 + m + 1 = b, r = m + 1 = k, \lambda = 1$ can be constructed provided m is a prime or a prime power. When m is *not* a prime or a prime power, this family of BIB designs does not exist for $m = 6, 14, 21, 22, 30, 33, 38, 42, 46$. Since the above family designs coexists with another family of BIB designs with parameters $v = m^2, b = m^2 + m, r + m + 1, k = m, \lambda = 1$, designs of the latter family also do not exist for the above values of m. For more on the existence of BIB designs, see Hanani (1961, 1975), Street and Street (1987, Chapters 12 and 13) and Abel and Greig (2007).

3.5 Some Generalizations of BIB designs

Kiefer (1958) introduced a class of designs, called balanced block (BB) designs as a generalization of BIB designs. As we shall see in Chapter 6, balanced block designs have strong optimality properties. A formal definition of BB designs, as given by Kiefer (1958) follows.

Definition 3.5.1 *A block design d with v treatments, b blocks, each of size k and incidence matrix $N_d = (n_{dij})$ is called a balanced block design if*
(i) $\sum_{j=1}^{b} n_{dij} = r$, *for* $1 \leq i \leq v$;
(ii) $\sum_{j=1}^{b} n_{dij} n_{dmj} = \lambda$, *for* $i \neq m$, $1 \leq i, m \leq v$;
(iii) $|n_{dij} - k/v| < 1$.

Das and Dey (1989) showed that for $v \geq 3$, condition (i) in Definition 3.5.1 is redundant. To describe this result, we first need the notion of a *generalized binary* design. A block design d is called a generalized binary design if N_d has only two distinct integral entries, x and $y = x+1$. Clearly, the usual binary design is a special case of a generalized binary design with $x = 0$. We now have the following result which is easy to prove.

Lemma 3.5.1 *Consider a generalized binary block design d with v treatments, b blocks and block size k. Then, $x = [k/v]$, where $[m]$ is the largest integer not exceeding m.*

We next have the following result.

Theorem 3.5.1 *Consider a binary or a generalized binary design d with $v \geq 3$ treatments, b blocks, block size k and incidence matrix $N_d = (n_{dij})$ such that*

$$\sum_{j=1}^{b} n_{dij} n_{dmj} = \lambda, \text{ for all } i \neq m, \quad 1 \leq i, m \leq v. \tag{3.5.1}$$

Then,

$$\sum_{j=1}^{b} n_{dij} = r, \text{ for } 1 \leq i \leq v, \tag{3.5.2}$$

where r and λ are some positive integers.

Proof. For any proper block design d with v treatments, b blocks each of size k and incidence matrix $N_d = (n_{dij})$,

$$N_d N_d' \mathbf{1}_v = k \boldsymbol{r}_d. \tag{3.5.3}$$

From our assumptions on the design d under consideration, the ith row sum of $N_d N_d'$ is

$$s_{di} + \lambda(v-1), \tag{3.5.4}$$

where $s_{di} = \sum_{j=1}^{b} n_{dij}^2$. Observe that for a binary design, $s_{di} = r_{di}$, $1 \leq i \leq v$. Thus for binary designs, the result follows by comparing the right sides of (3.5.3) and (3.5.4). Now let d be generalized binary with $v > 2$ and let $t = [k/v]$. Suppose m_i is the number of blocks in d in which the ith treatment appears $t+1$ times. Then, we have

$$r_{di} = m_i + bt \text{ and } s_{di} = bt^2 + (2t+1)m_i. \tag{3.5.5}$$

Equating the right side of (3.5.3) and (3.5.4) for a generalized binary design and using (3.5.5), we have

$$m_i(k - 2t - 1) = \lambda(v-1) - bkt + bt^2. \tag{3.5.6}$$

Since the right side of (3.5.6) is independent of i, m_i and hence, r_{di} is also independent of i. □

We next consider another generalization of BIB designs, namely balanced n-ary designs. Such designs were initially considered by Tocher (1952) for the special case $n = 3$ and were called balanced ternary designs. A definition of a balanced n-ary design follows.

Definition 3.5.2 *An equireplicate, proper block design with v treatments, b blocks, block size k, replication r and incidence matrix $N_d = (n_{dij})$ is called a balanced n-ary design if for some integer $n \geq 3$, (i) $n_{dij} \in \{0, 1, \ldots, n-1\}$ and (ii) $\sum_{j=1}^{b} n_{dij} n_{dmj} = \lambda$ for all $i \neq m$, where $\lambda > 0$ is an integer.*

As an example, consider the following block design with $v = 4$ treatments and $b = 12$ blocks:

$$\begin{array}{llll} (1,1,2,3); & (1,2,2,4); & (1,1,2,4); & (1,2,2,3); \\ (1,3,3,4); & (2,3,4,4); & (1,3,4,4); & (2,3,3,4); \\ (1,1,3,4); & (1,2,3,3); & (1,2,4,4); & (2,2,3,4). \end{array}$$

Then, it is easy to verify that the above design is a balanced n-ary design with $n = 3$. Balanced n-ary designs have been studied by several authors and construction methods of such designs may be found in Murty and Das (1968), Dey (1970) and Saha and Dey (1973). Balanced n-ary designs with $n \geq 3$ are however, not very appealing as statistical designs. This is because if $k = v$, then one can use a more efficient randomized complete block design (which is orthogonal) in preference to a non-orthogonal balanced n-ary design. Even if $k < v$, a binary design (like a BIB design) is almost always more efficient than a comparable balanced n-ary design with $n > 2$.

A third generalization of BIB designs are known by the name t-designs or tactical configurations. These are defined below.

Definition 3.5.3 *Given a set $\mathcal{T} = \{1, 2, \ldots, v\}$ of v treatments and positive integers k, t ($t \leq k \leq v$) and δ, a t-design $\mathcal{H}(\delta, t, k, v)$ is defined to be a system of blocks (subsets of $\mathcal{T}$), each of size k, such that each subset of t elements is contained in exactly δ blocks.*

Clearly, a 2-design is a BIB design. Note that a t-design is also an s-design for $0 < s < t$. Let d be a t-design and let $s < t$ be a positive integer. If λ_t (respectively, λ_s) denotes the number of blocks containing a set of t (respectively, s) treatments, then the following result holds (see e.g., Beth, Jungnickel and Lenz (1993, p. 29)):

$$\lambda_s \binom{k-s}{t-s} = \lambda_t \binom{v-s}{t-s}. \tag{3.5.7}$$

Designs with $t = 3$ are also called *doubly balanced* incomplete block designs (Calvin, 1954). For instance, the design given in Example 3.4.2

is a 3-design with $v = 8, b = 14, k = 4, r = 7, \lambda_2 = 3, \lambda_3 = 1$. If d is a symmetric BIB design with parameters $v = 4u - 1 = b, r = 2u - 1 = k, \lambda = u - 1$ (see Section 3.4), then its complementary design, $\bar{d}$ is a BIB design with parameters $\bar{v} = 4u - 1 = \bar{b}, \bar{r} = 2u = \bar{k}, \bar{\lambda} = u$. Suppose d^* is a design obtained by augmenting each block of d by a new treatment, say ∞. Then the collection of blocks of d^* and $\bar{d}$ form a 3-design (called a Hadamard 3-design) with parameters $v_3 = 4u, b_3 = 8u - 2, r_3 = 4u - 1, k_3 = 2u, \lambda_2 = 2u - 1, \lambda_3 = u - 1$.

A t-design is said to be trivial if $k = v - 1$. It was shown by Raghavarao (1970) that for a non-trivial t-design, the inequality

$$b \geq (t - 1)(v - t + 2) \tag{3.5.8}$$

holds. From (3.5.8), we get the familiar Fisher's inequality for a BIB design ($t = 2$). Similarly, for a doubly balanced incomplete block design (i.e., $t = 3$), the inequality $b \geq 2(v - 1)$ holds. Under certain conditions, a sharper inequality than (3.5.8) was obtained by Dey and Saha (1974). See also Raychaudhuri (1975) and Wilson (1983) in this connection. For more details on t-designs including an extensive list of references, we refer to Hedayat and Kageyama (1980) and Kageyama and Hedayat (1983). See also Khosrovshahi and Laue (2007).

3.6 Construction of Variance- and Efficiency-balanced Designs

Among the class of equireplicate, proper and binary designs, a BIB design, if existent, is the only variance-balanced design. The balanced n-ary designs, $n \geq 3$ defined above are equireplicate, proper, nonbinary variance-balanced designs. If we enlarge the class of designs to include non-equireplicate, nonbinary and non-proper designs, then there exist other incomplete block designs which are variance-balanced. The construction of such variance-balanced designs has been considered by several authors (see e.g., Kulshreshtha, Dey and Saha (1972), Hedayat and Federer (1974), Kageyama (1976, 1988a,b), Khatri (1982), Gupta and Jones (1983), Agarwal and Kumar (1984, 1986), Mukerjee and Kageyama (1985), Jones, Sinha and Kageyama (1987) and Hedayat and Stufken (1989)). We present here a selection of such methods.

Kulshreshtha, Dey and Saha (1972) were probably the first to present a general method of construction of variance-balanced designs with two unequal block sizes and we now describe their construction. Let d_1

be a BIB design with parameters v, b, r, k, λ, where the treatments are labeled $1, 2, \ldots, v$. Augment each block of d_1 by $k^* \geq 1$ replicates of a new treatment, labeled 0. Thus, we have a design involving $v + 1$ treatments, say d_{10}, in b blocks, each of size $k_1 = k + k^*$. Next, let d_2 be another BIB design with parameters v, b', r', k', λ'. Repeat the design d_{10}, n times and the design d_2, m times, where m, n are integers, to be determined. Kulshreshtha *et al.* (1972) proved the following result, whose proof is left as an exercise.

Theorem 3.6.1 *The design described above, involving $v+1$ treatments and $nb + mb'$ blocks is a variance-balanced design whenever m, n are such that $\frac{m}{n} = \frac{k'(k^*r - \lambda)}{\lambda'(k + k^*)}$. The treatment with label 0 has replication $r_0 = nbk^*$ while the ith among the remaining treatments have replication $r_i = nr + mr'(1 \leq i \leq v)$ and the two block sizes are $k_1 = k + k^*$ and $k_2 = k'$. Also, the design is non-binary for $k^* > 1$ and binary otherwise.*

The next result provides another method of constructing non-equireplicate, non-proper variance-balanced designs.

Theorem 3.6.2 *Let t be an odd integer and $s = (t + 1)/2$. Consider a design d with incidence matrix N_d given by*

$$N_d = \begin{bmatrix} \mathbf{0}' & \mathbf{1}_t' & 1 \\ \mathbf{1}_s' & \mathbf{0}' & 1 \\ J_{ts} & I_t & \mathbf{0} \end{bmatrix}.$$

Then, d is a variance-balanced design.

The following result also uses a BIB design and a variance-balanced design to construct another variance-balanced design.

Theorem 3.6.3 *Let N_1 be the incidence matrix of a BIB design with parameters $v_1, b_1, r_1, k_1, \lambda_1$ and N_2, that of a variance balanced design with v_1 treatments, b_2 blocks, replication vector $\boldsymbol{r}$ and block size vector $\boldsymbol{k}$. Then*

$$N = \begin{bmatrix} \mathbf{1}_{mb_1}' & \mathbf{0}' \\ \mathbf{1}_m' \otimes N_1 & \mathbf{1}_n' \otimes N_2 \end{bmatrix}$$

is the incidence matrix of a variance-balanced design whenever for positive integers m, n, $\frac{n}{m} = \frac{v_1(v_1 - 1)(r_1 - \lambda_1)}{(n' - b_2)(k_1 + 1)}$, where n' is the number of experimental units in the initial variance balanced design.

The proofs of the above results consist in verifying that the C-matrix of the final design is of the form $C = \theta(I_v - v^{-1}J_v)$ for some positive scalar θ, where v is the number of treatments in the final design. For a collection of several other methods of construction of variance-balanced designs with possibly unequal replications and unequal block sizes, see Caliński and Kageyama (2000).

A wide variety of methods of construction of efficiency-balanced incomplete block designs are available in the literature. In the following, we discuss some of these methods. We first have the following result.

Theorem 3.6.4 *For an integer $t \geq 1$, let*

$$N_d = \begin{bmatrix} (J_{t+1} - I_{t+1}) \otimes \mathbf{1}_t' & I_{t+1} \otimes \mathbf{1}_t' & J_{t+1} \\ J_{t,t(t+1)} & \mathbf{1}_{t+1}' \otimes I_t & \mathbf{0} \end{bmatrix}.$$

Then, N_d is the incidence matrix of an equireplicate efficiency-balanced incomplete block design with parameters $v = 2t+1, b = (t+1)(2t+1), r = (t+1)^2, \boldsymbol{k} = (2t\mathbf{1}_{t^2+t}', 2\mathbf{1}_{t^2+t}', (t+1)\mathbf{1}_{t+1}')'$ and efficiency factor $\epsilon = \dfrac{2t+1}{2t+2}$.

Proof. Follows by invoking Corollary 2.3.2. □

The next result uses an efficiency-balanced design to derive another efficiency-balanced design.

Theorem 3.6.5 *Let N_1 be the $v_1 \times b_1$ incidence matrix of an equireplicate, proper efficiency-balanced design d_1 with efficiency factor $1 - \epsilon_1$ and let the block size and replication of d_1 be k_1 and r_1 respectively. Consider the matrix N_d given by*

$$N_d = \begin{bmatrix} N_1 & J_{v_1 b_1} - N_1 \\ c\mathbf{1}_{b_1}' & \mathbf{0}' \end{bmatrix}.$$

Then, N_d is the incidence matrix of an efficiency-balanced design with parameters $v = v_1 + 1, b = 2b_1, \boldsymbol{r} = b_1(\mathbf{1}_{v_1}', c)', \boldsymbol{k} = ((k_1 + c)\mathbf{1}_b', (v_1 - k_1)\mathbf{1}_b')'$, provided $c = \dfrac{k_1^2 v_1 (1-\epsilon_1)}{(v_1 - k_1)^2 - k_1^2(1-\epsilon_1)}$ is a positive integer.

For more methods of construction of efficiency-balanced block designs, see Caliński and Kageyama (2000).

3.7 Nested BIB Designs

In some experimental situations, there might exist more sources of variation that can be controlled by ordinary blocking and there might exist plausible relationships among several sources of variation. An important relationship that is often encountered in practice is that of nesting. If A and B are two blocking factors then A is said to nest B if units in two different blocks according to attribute A are in two different blocks according to the attribute B. An equivalent way of stating this is that each block according to A is a union of blocks according to B. An early example of such nesting can be found in Preece (1967), which is described as follows: Suppose the half-leaves of a plant form the experimental units, on which a number of treatments are to be tested. The treatments for instance could be inoculations with sap from tobacco plants infected with a certain virus. Suppose the number of treatments is more than the number of available half-leaves per plant. Clearly, one source of variation is due to the variability among the plants. Further, leaves within a plant might exhibit variation due to their location on an upper, middle or lower branch of the same plant. Therefore, leaves within plants are a nested nuisance factor, the nesting being within the plants. The half-leaves being the experimental units, there are two systems of blocks, leaves (which may be called sub-blocks) being nested within plants. For some more examples and discussion of nesting, we refer to Srivastava (1978, 1981) and Morgan (1996). In this section, we describe some aspects of nested incomplete block designs, with special emphasis on nested balanced incomplete block designs. A nested block design has the following structure: there are b_1 blocks each of size b_2k and each of these blocks nests b_2 blocks of size k each. Thus, the blocking factor A has b_1 levels, the nested blocking factor (sub-block) B has b_1b_2 levels and the total number of experimental units is $n = b_1b_2k$. Let Y_{ijl} be the response from plot (unit) l in sub-block j of block i. A model suitable for the observations is given by

$$Y_{ijl} = \mu + \beta_i^{(1)} + \beta_{ij}^{(2)} + \tau_{ijl} + \epsilon_{ijl}, \tag{3.7.1}$$

where μ is an overall mean, $\beta_i^{(1)}$ is the effect of block i, $\beta_{ij}^{(2)}$, the effect of the sub-block j in the block i, τ_{ijl} is the effect of the treatment assigned to the unit (i, j, l) and ϵ_{ijl} is a random error term, the error terms being assumed to be uncorrelated random variables with zero means and constant variance. Let $\boldsymbol{Y}$ be the $n \times 1$ vector of the quantities

$\{Y_{ijl}\}$ arranged in lexicographic order. Then the model (3.7.1) can be written in matrix notation as

$$Y = \mu \mathbf{1}_n + L_1\boldsymbol{\beta}^{(1)} + L_2\boldsymbol{\beta}^{(2)} + A\boldsymbol{\tau} + \boldsymbol{\epsilon}, \tag{3.7.2}$$

where $L_1 = I_{b_1} \otimes \mathbf{1}_{b_2k}, L_2 = I_{b_1b_2} \otimes \mathbf{1}_k$, A is the $n \times v$ design matrix with elements 0 and 1, indicating the assignment of treatments to the experimental units, $\boldsymbol{\beta}^{(1)}$ and $\boldsymbol{\beta}^{(2)}$ are $b_1 \times 1$ and $b_1b_2 \times 1$ vectors with elements $\beta_i^{(1)}$ and $\beta_{ij}^{(2)}$ respectively, $\boldsymbol{\tau}$ is the $v \times 1$ vector of treatment effects and $\boldsymbol{\epsilon}$ is the vector of random error components.

As in the case of intra-block analysis of ordinary block designs, if only linear combinations of contrasts of responses from within blocks are allowed, then one is performing a "within block" or *bottom stratum* analysis. In contrast, as in the recovery of inter-block information in the context of ordinary block designs, if contrasts among block totals are used for estimation, then one is performing what is called the *full* analysis. The bottom stratum information matrix (the now familiar C-matrix) under a nested block design d is given by (see, Morgan (1996) and Remark 2.2.1)

$$C_d = A_d'(I - \mathrm{pr}(L))A_d, \tag{3.7.3}$$

where A_d is the matrix A for the design d, $L = (\mathbf{1}_n, L_1, L_2)$ and $\mathrm{pr}(L)$, as in Section A.1 of the Appendix, is the matrix which projects onto the column space of L. Since the column spans of L and L_2 are the same, $\mathrm{pr}(L) = \mathrm{pr}(L_2)$ and thus,

$$C_d = A_d'(I - \mathrm{pr}(L_2))A_d. \tag{3.7.4}$$

The matrix C_d in (3.7.4) is the same as the C-matrix that would be obtained under the simpler model $Y = \mu\mathbf{1} + L_2\boldsymbol{\beta}^{(2)} + A_d\boldsymbol{\tau} + \boldsymbol{\epsilon}$, which implies that in the bottom stratum analysis, the block factor has no role to play. The full analysis, in contrast, has a C-matrix that depends on both the block and the sub-block C-matrices. We do not elaborate further on these issues but refer to Morgan (1996) for an excellent description of the methods involved and other related issues. See also Cheng (1986) for a discussion on this aspect in the context of a related class of designs.

We now focus attention on nested balanced incomplete block (NBIB) designs, introduced by Preece (1967), who fully outlined their analysis and provided a table of small NBIB designs.

Definition 3.7.1 *A nested balanced incomplete block (NBIB) design is an incomplete block design involving v treatments, each replicated r times and having two systems of blocks such that*

(i) the second system is nested within the first, with each block from the first system (called "blocks") containing exactly m blocks of the second system (called "sub-blocks");
(ii) ignoring the sub-blocks leaves a BIB design with parameters $v, b_1, r, k_1, \lambda_1$;
(iii) ignoring the blocks leaves a BIB design with parameters $v, b_2, r, k_2, \lambda_2$.

Here is an example of an NBIB design.

Example 3.7.1 Let $v = 5, b_1 = 5, b_2 = 10, r = 4, k_1 = 4, k_2 = 2$. Then the following is an NBIB design with the stated parameters. Here, the square brackets include blocks, the parentheses include the sub-blocks and the treatment labels are $0, 1, \ldots, 4$.

$$[(1,4),(2,3)];\ [(2,0),(3,4)];\ [(3,1),(4,0)];\ [(4,2),(0,1)];\ [(0,3),(1,2)].$$

Here, $m = 2, \lambda_1 = 3$ and $\lambda_2 = 1$.

The parameters of an NBIB design satisfy the following necessary conditions:

$$\begin{aligned} vr &= b_1k_1 = b_1mk_2 = b_2k_2, \\ \lambda_1(v-1) &= r(k_1-1), \lambda_2(v-1) = r(k_2-1). \end{aligned} \tag{3.7.5}$$

It follows then that

$$(v-1)(\lambda_1 - m\lambda_2) = r(m-1). \tag{3.7.6}$$

An NBIB design will be denoted by NBIB (v, b_1, b_2, k). Methods of construction of NBIB designs has been studied by various authors, including Preece (1967), Jimbo and Kuriki (1983), Bailey, Goldrei and Holt (1984), Dey, Das and Banerjee (1986), Iqbal (1991), Jimbo (1993) and Kageyama and Miao (1998). We present below a selection of these methods. For proofs of these results, the original sources may be referred to.

Theorem 3.7.1 (Jimbo and Kuriki (1983)). *Consider a BIB design with parameters $v^*, b^*, k^* = v$ and an NBIB (v, b_1, b_2, k). If the NBIB design is written using the treatments of each block of the BIB design, then the resulting design is an NBIB (v^*, b_1b^*, b_2, k).*

Note that a resolvable incomplete block design can be regarded as a special case of a nested incomplete block designs, with the blocks being complete. In view of this fact, if a resolvable BIB design is taken as a particular case of an NBIB design in Theorem 3.7.1, then one gets an NBIB design obtained by Dey *et al.* (1986).

Theorem 3.7.2 *The existence of an NBIB design* (v, b_1, b_2, k) *and of a resolvable BIB design with parameters* $v^* = b_2, b^*, k^*$ *implies the existence of an NBIB*$(v, b_1 b^*, k^*, k)$.

The method of differences has also been used for the construction of NBIB designs. To describe one such result due to Jimbo and Kuriki (1983), we first introduce some notation. Let v be a prime or a prime power and x, a primitive element of $GF(v)$. For any m that divides $v-1$, let $H_{m,0} = (x^0, x^m, x^{2m}, \ldots, x^{v-1-m})'$ and for $0 \leq i \leq m-1$, let $H_{m,i} = x^i H_{m,0}$. Furthermore, let $S_m = (x^0, x, \ldots, x^{m-1})$. Then, we have the following result due to Jimbo and Kuriki (1983), where sub-blocks within a block are displayed separated by bars.

Theorem 3.7.3 *Let* $v = mu + 1$ *be a prime or a prime power and for* $1 \leq i \leq n$, *let* L_i *be mutually disjoint subsets of* S_m, *each subset being of cardinality* s *and these subsets being written as* $s \times 1$ *vectors. Also, let* $A_i = L_i \otimes H_{m,i}$. *Then the* m *initial blocks*

$$B_j = x^{j-1} \left[\begin{array}{c} A_1 \\ \hline A_2 \\ \hline \vdots \\ \hline A_n \end{array} \right], 1 \leq j \leq m$$

on development yield an NBIB(v, mv, n, us). *Furthermore, if* m *is even and* u *is odd, the initial blocks* $B_1, \ldots, B_{m/2}$ *generate an NBIB*$(v, mv/2, n, us)$.

For some other constructions based on the method of differences, see Jimbo and Kuriki (1983) and Dey *et al.* (1986). For an excellent description of construction and other aspects of nested balanced incomplete block designs, we refer to Morgan, Preece and Rees (2001), who discuss all known methods of construction and provide extensive tables of designs in the parametric range $v \le 16$, $r \le 30$. A shorter table of designs with $v \le 14$, $r \le 30$ is available in Morgan (1996).

A generalization of nested block designs was introduced and studied by Singh and Dey (1979). These designs are called block designs with nested rows and columns and involve two nested nuisance factors. Such designs have been studied in greater detail by several authors, including Agrawal and Prasad (1982, 1983), Cheng (1986), Sreenath (1989, 1991), Uddin and Morgan (1990, 1991) and Mukerjee and Gupta (1991b). For more details, the above references may be consulted.

3.8 Exercises

3.1. Prove that for a BIB design (v, b, r, k, λ), the Fisher's inequality is equivalent to the inequality $b \ge v + r - k$.

3.2. Prove that for a BIB design (v, b, r, k, λ), the inequality $b \ge v+r-1$ is equivalent to $r \ge k + \lambda$.

3.3. Let d be a resolvable BIB design with incidence matrix N_d. Obtain an upper bound for the rank of N_d and hence obtain the inequality $b \ge v + r - 1$.

3.4. Let N_d be the incidence matrix of a symmetric BIB design d. By obtaining the inverse of $N_d N_d'$, show that any two blocks of d intersect in λ treatments.

3.5. Let y_i be the number of treatments common between a given block of a BIB design and the ith of the remaining blocks. By computing the variance of the y_i-values, give an alternative proof of the Fisher's inequality.

3.6. Let d be a BIB design, $\bar{d}$, its complementary design and d_0 be a design (with possibly unequal block sizes) defined as $d_0 = d \cup \bar{d}$. Show that in d_0, each triplet of treatments occurs together in $b - 3r + 3\lambda$ blocks. Hence prove that for a BIB design, the inequality $b \ge 3(r - \lambda)$ holds.

3.7. For a symmetric BIB design, express the adjusted block sum of squares in terms of the quantities $\{W_i\}$, where W_i is as defined in (3.3.18).

3.8. Show that for a BIB design belonging to Family(A), v and k are related by the following identity: $2k = v \pm v^{\frac{1}{2}}$.

3.9. For a doubly balanced incomplete block design with v treatments and b blocks, prove that $b \geq 2(v-1)$.

3.10. Let d be a resolvable BIB design satisfying $b = 2r$. Show that such a design is a 3-design. Furthermore, show that for such a design, $3\lambda - r$ is an even integer.

3.11. Give an example of a design to show that the conclusion in Theorem 3.5.1 does not hold for $v = 2$.

3.12. Let N_d be the incidence matrix of a BIB design belonging to Family(A) and suppose that M is a matrix obtained fromN_d by replacing the zeros in N_d by -1. Show that the rows of M are mutually orthogonal.

3.13. Let $u \geq 2$ be an integer. Show that BIB designs with the following sets of parameters exist:

v	b	r	k	λ
$2^u - 1$	$2^u - 1$	$2^{u-1} - 1$	$2^{u-1} - 1$	$2^{u-2} - 1$
$2^u - 1$	$2^u - 1$	2^{u-1}	2^{u-1}	2^{u-2}
2^{u-1}	$2(2^{u-1} - 1)$	$2^{u-1} - 1$	2^{u-2}	$2^{u-2} - 1$
$2^{u-1} - 1$	$2(2^{u-1} - 1)$	$2(2^{u-2} - 1)$	$2^{u-2} - 1$	$2(2^{u-3} - 1)$
$2^{u-1} - 1$	$2(2^{u-1} - 1)$	2^{u-1}	2^{u-2}	2^{u-2}.

3.14. Let N_d be the incidence matrix of a BIB design d with parameters given by (3.4.21) and $\bar{N}_d = J_{4u-1} - N_d$. Show that the matrix N_1 given by

$$N_1 = \begin{bmatrix} N_d & \bar{N}_d & \mathbf{0} \\ \mathbf{1}' & \mathbf{0}' & 0 \\ N_d & \bar{N}_d & \mathbf{1} \end{bmatrix}$$

is the incidence matrix of a BIB design.

3.15. Let $v = 2k$ and suppose that $v - 1$ is a prime or a prime power. Also, let x be a primitive element of $GF(v-1)$. Show that the initial blocks $(0, x^i, x^{i+2}, \ldots, x^{i+2k-4}), (\infty, x^{i+1}, x^{i+3}, \ldots, x^{i+2k-3})$, $i = 0, 1$, provide a solution for a BIB design. Determine the parameters of the design.

3.16. Let m be a prime or a prime power. Show that the following families of BIB designs coexist:
(i) $v = m^2, b = m^2 + m, r = m + 1, k = m, \lambda = 1$;
(ii) $v = m^2 + m + 1 = b, r = m + 1 = k, \lambda = 1$.

3.17. Let x be a primitive element of the Galois field $GF(2^4)$. Show that a solution of the BIB design with parameters $v = 16 = b, r = 6 = k, \lambda = 2$ is provided by the initial block $(0, x^0, x^3, x^6, x^9, x^{12})$.

3.18. Obtain a solution of the BIB design with parameters $v = 27 = b, r = 13 = k, \lambda = 6$.

3.19. Give a method of construction of a family of BIB designs with parameters $v = 4u, b = 8u - 2, r = 4u - 1, k = 2u, \lambda = 2u - 1$ where u is a positive integer.

3.20. Using an appropriate finite geometry, obtain a solution of the BIB design with parameters $v = 40 = b, r = 13 = k, \lambda = 4$.

3.21. Suppose d is a BIB design (v, b, r, k, λ) where $r = 2k + 1$ and $\lambda = 1$. Show that the existence of d implies the existence of a symmetric BIB design with $(4k^2 - 1)$ treatments, replication $2k^2$ and pairwise concurrence parameter equal to k^2.

3.22. A $v \times b$ matrix M with entries $\pm 1, 0$ is called a generalized balanced matrix if

(i) the inner product of any two distinct rows of M is a constant, say μ and,

(ii) when the -1's in M are replaced by $+1$'s, the resultant matrix becomes the incidence matrix of a BIB design with parameters v, b, r, k, λ.

Show that a necessary condition for the existence of a generalized balanced matrix is that $\lambda \equiv \mu \pmod 2$.

3.23. Let d be a symmetric BIB design with incidence matrix N_d and block size k. Show that N_d can be expressed as a sum of k permutation matrices (a square matrix with a single entry of unity in each row and each column and all other elements zero is called a permutation matrix).

3.24. Let d_1 be an efficiency-balanced design on v treatments and suppose the treatment labels are $a_1, a_2, \ldots, a_v$. Partition the treatment labels into s disjoint sets and let d be a design by replacing the labels belonging to the same set by a new treatment, so that now there are only s treatments in d. Examine whether d is also an efficiency-balanced design.

3.25. Consider the designs d_1 and d_2, each involving $v = 4$ treatments, $b = 10$ blocks, block size $k = 3$ and replication vector $\boldsymbol{r} = (6, 6, 6, 12)'$:

$$
\begin{aligned}
d_1 \quad &: \quad (1,2,3); (1,2,4); (1,2,4); (1,3,4); (1,3,4); (2,3,4); (2,3,4); \\
&\quad\;\; (1,4,4); (2,4,4); (3,4,4). \\
d_2 \quad &: \quad (1,2,3); (1,2,4); (1,3,4); (2,3,4); (1,2,3); (1,2,4); (1,3,4); \\
&\quad\;\; ((2,3,4); (4,4,4); (4,4,4).
\end{aligned}
$$

Show that d_1 is efficiency-balanced and d_2 is variance-balanced. Which of the two designs would you prefer based on the average variance of the BLUEs of all elementary treatment contrasts?

3.26. For a positive integer s, let

$$N_d = \begin{bmatrix} \mathbf{1}'_s & 0 \\ I_s & \mathbf{1}_s \end{bmatrix}.$$

Show that N_d is the incidence matrix of an efficiency-balanced design and determine its parameters.

3.27. Give an example of an NBIB design constructed using Theorem 3.7.1.

Chapter 4

Partially Balanced Designs

4.1 Introduction

In Chapter 3, we have discussed the properties, analysis and construction of variance- and efficiency-balanced designs with emphasis on BIB designs. In the class of equireplicate, proper and binary designs, the BIB designs are the only variance-balanced designs. It will be seen later in this book that BIB designs, whenever existent, have strong optimality properties. If we restrict our attention to only equireplicate, proper and binary designs, then as observed in Chapter 3 (Section 3.4.4), BIB designs do not exist for every combination of the parameters, satisfying the necessary conditions. Even if a BIB design exists for a given value of v, the number of treatments and k, the block size, it might sometimes require too many replications, resulting in the increased size of the experiment. In situations where the size of the experiment is limited due to cost and other considerations, one might have to sacrifice the property of variance-balance and look for designs that are available with reasonable number of experimental units. Quite naturally, such designs might be called unbalanced or, *partially* balanced. A variety of partially balanced designs are now available, among which the most important ones are the *partially balanced incomplete block* (PBIB) designs, introduced by Bose and Nair (1939). Unlike the BIB designs, in a PBIB design, the variance of the best linear unbiased estimator of an elementary treatment contrast is *not* a constant, justifying the name "partially balanced". The original definition of PBIB designs was modified by Nair and Rao (1942b) and the current definition of such designs is based on an abstract relation, called an *association scheme*, a notion introduced by Bose and Shimamoto (1952). The literature on PBIB designs is extremely rich. In Sections 4.2–4.5, we present a selection of the vast amount of results in the area of PBIB designs. The analysis of PBIB designs is discussed briefly in Section 4.6. In sections 4.7–4.11, several other unbalanced designs are considered. In Section 4.7, we discuss lattice or, quasi-factorial designs. Cyclic designs, linked block designs and C-designs, which are all in general not balanced, are covered in Sections 4.8, 4.9 and 4.10, respectively. An important class of resolvable incomplete block designs, called α designs are discussed in Section 4.11.

4.2 Introducing PBIB Designs

The definition of a PBIB design is based on the notion of an association scheme, which we define now.

Definition 4.2.1 *A relationship defined on v symbols (or, treatments) is called an m-class association scheme ($m \geq 2$) if the following conditions hold:*
(i) A pair of treatments θ, ϕ are either 1st, 2nd, ..., or mth associates, the relation of association being symmetric, i.e., if θ is the ith associate of ϕ, then so is ϕ of θ;
(ii) for a given treatment θ, the number of ith associates ($1 \leq i \leq m$) of θ is n_i, where the integer n_i is independent of the given treatment, i.e., θ;
(iii) given a pair of treatments θ, ϕ that are mutually ith associates, the number of treatments that are simultaneously jth associate of θ and sth associate of ϕ is p^i_{js}, where this number does not depend on the particular pair of treatments chosen as long as they are mutually ith associates.

The integers v, n_i, p^i_{js} $(1 \leq i, j, s \leq m)$ are called the parameters of the m-class association scheme.

Lemma 4.2.1 *The following relations hold among the parameters of an m-class association scheme:*

$$\sum_{i=1}^{m} n_i = v - 1; \ \sum_{s=1}^{m} p^i_{js} = n_j - \delta_{ij}; \ n_i p^i_{js} = n_j p^j_{is}, \tag{4.2.1}$$

where $\delta_{ij} = 1$, if $i = j$, and is zero, otherwise.

Proof. The first relation in (4.2.1) is trivially true. Suppose θ and ϕ are a pair of treatments that are mutually ith associates. Then the sth associates of θ $(1 \leq s \leq m)$ must cover all the jth associates of ϕ, these being n_j in number. Therefore, $\sum_{s=1}^{m} p_{js}^{i} = n_j$ for $j \neq i$. When $j = i$, θ itself is one of the jth associates of ϕ and, thus arguing as before, we have $\sum_{s=1}^{m} p_{is}^{i} = n_i - 1$, which proves the second relation in (4.2.1).

To see the truth of the third relation in (4.2.1), let $S_i(S_j)$ be the set of ith (jth) associates of a treatment θ. Any treatment in S_i has p_{js}^{i} sth associates in S_j and similarly, any treatment in S_j has p_{is}^{j} sth associates in S_i. Forming pairs from S_i and S_j, on one side we have $n_i p_{js}^{i}$ and on the other, we have $n_j p_{is}^{j}$. Since they represent the same quantity, they must be equal, giving the third identity in (4.2.1). □

It is sometimes convenient to introduce a zero-th associate class by defining each treatment to be its own zero-th associate and of no other treatment. Clearly then we have

$$n_0 = 1, \quad p_{ij}^{0} = n_i \delta_{ij}, \quad p_{0s}^{i} = \delta_{is}. \tag{4.2.2}$$

In view of this, the relations in (4.2.1) can be restated as

$$\sum_{i=0}^{m} n_i = v; \; \sum_{s=0}^{m} p_{js}^{i} = n_j; \; n_i p_{js}^{i} = n_j p_{is}^{j}, \quad 0 \leq i, j, s \leq m. \tag{4.2.3}$$

We are now in a position to define a PBIB design with m associate classes.

Definition 4.2.2 *Given an association scheme $\mathcal{A}$ with v treatments and $m(\geq 2)$ classes, we have a PBIB design based on $\mathcal{A}$ if it is possible to arrange the treatments into b blocks such that*
(i) each block has $k(< v)$ distinct treatments,
(ii) each treatment appears in r blocks,
(iii) if the treatments θ and ϕ are mutually ith associates in $\mathcal{A}$, then these appear together in λ_i blocks where λ_i does not depend on the pair (θ, ϕ) as long as they are mutually ith associates $(1 \leq i \leq m)$. Also, not all λ_i's are equal.

The integers v, b, r, k, λ_i are called the parameters of the PBIB design. Clearly, the definition of an m-associate PBIB design is based on the existence of an association scheme on m classes and thus, if for some values of v, n_i, p_{js}^{i}, m, there is no association scheme with m classes then there is no PBIB design with m-associate classes based on the scheme. For example, it is known (Mesner (1965)) that there is no two-class association scheme ($m = 2$) with v treatments when v is a prime number of the form $4u + 3$. Thus, there does not exist any two-associate PBIB design with $v = 7, 11, 19$ etc. We next have the following result connecting the parameters of a PBIB design.

Lemma 4.2.2 *For a PBIB design d with parameters v, b, r, k, λ_i based on an association scheme with parameters v, n_i, p^i_{js}, the following are true:*

$$(i)\ vr = bk; \quad (ii) \sum_{i=1}^{m} n_i \lambda_i = r(k-1). \tag{4.2.4}$$

Proof. Relation (i) is trivial. To see the truth of (ii), let N_d be the incidence matrix of the design d. Then,

$$N_d N_d' \mathbf{1}_v = \left\{ r + \sum_{i=1}^{m} n_i \lambda_i \right\} \mathbf{1}_v. \tag{4.2.5}$$

Also,

$$N_d N_d' \mathbf{1}_v = N_d(N_d' \mathbf{1}_v) = k N_d \mathbf{1}_b = rk \mathbf{1}_v. \tag{4.2.6}$$

Comparing (4.2.5) and (4.2.6) we get the required result. □

4.3 The Algebra of Association Matrices

The combinatorial properties of association schemes and PBIB designs based on them can be conveniently studied through *association matrices*, introduced by Bose and Mesner (1959). Consider an association scheme $\mathcal{A}$ with parameters v, n_i, p^i_{js}, $0 \le i, j, s \le m$. For $0 \le i \le m$, the ith *association matrix* $B_i = (b^i_{xy})$ is a symmetric matrix of order v with

$$\begin{aligned} b^i_{xy} &= 1, \text{ if } x \text{ and } y \text{ are mutually } i\text{th associates} \\ &= 0, \text{ otherwise, } 1 \le x, y \le v. \end{aligned} \tag{4.3.1}$$

It follows then that $B_0 = I_v$ and for $1 \le i \le m$,

$$B_i \mathbf{1}_v = n_i \mathbf{1}_v. \tag{4.3.2}$$

Also, it is not hard to see that

$$\sum_{i=0}^{m} B_i = J_v \tag{4.3.3}$$

and the linear form $\sum_{i=0}^{m} c_i B_i = \mathbf{0}$ if and only if the scalars $c_0, c_1, \ldots, c_m$ are each equal to zero. Thus, the matrices B_i, $0 \le i \le m$, are linearly independent and linear functions of these matrices form a vector space of dimension $m+1$ with basis $\{B_0, B_1, \ldots, B_m\}$.

Consider the product of two association matrices. Since the (α, β)th entry of $B_j B_s$ is the number of treatments common between the jth associates of α and sth associates of β, we have

$$B_j B_s = \sum_{i=0}^{m} p^i_{js} B_i, \; 0 \leq j, s \leq m. \tag{4.3.4}$$

Since the association matrices are symmetric, we have

$$\begin{aligned} B_s B_j = B'_s B'_j &= (B_j B_s)' \\ &= \left(\sum_{i=0}^{m} p^i_{js} B_i\right)' = \sum_{i=0}^{m} p^i_{js} B'_i = \sum_{i=0} p^i_{js} B_i \\ &= B_j B_s. \end{aligned} \tag{4.3.5}$$

Hence, the association matrices commute under multiplication. Also, it follows then that

$$p^i_{js} = p^i_{sj}. \tag{4.3.6}$$

For $0 \leq i \leq m$, consider now the following square matrices, each of order $m+1$:

$$\mathcal{P}_i = (p^s_{ji}) = \begin{bmatrix} p^0_{0i} & p^1_{0i} & p^2_{0i} & \cdots & p^m_{0i} \\ p^0_{1i} & p^1_{1i} & p^2_{1i} & \cdots & p^m_{1i} \\ \vdots & & & & \\ p^0_{mi} & p^1_{mi} & p^2_{mi} & \cdots & p^m_{mi} \end{bmatrix}. \tag{4.3.7}$$

Since matrix multiplication is associative,

$$B_i(B_j B_s) = \sum_{u,t} p^u_{js} p^t_{iu} B_t = (B_i B_j) B_s = \sum_{u,t} p^u_{ij} p^t_{us} B_t.$$

Furthermore, since $B_0, B_1, \ldots, B_m$ are linearly independent, we have

$$\sum_{u} p^u_{js} p^t_{iu} = \sum_{u} p^u_{ij} p^t_{us}. \tag{4.3.8}$$

This means that

$$\mathcal{P}_j \mathcal{P}_s = \sum_{i=0}^{m} p^i_{js} \mathcal{P}_i. \tag{4.3.9}$$

Thus the matrices $\{\mathcal{P}_i\}$ multiply in the same way as the association matrices. Furthermore, the $\mathcal{P}_i$ matrices are linearly independent and form the basis of a vector space of dimension $m+1$. Note that these matrices combine in the same way as the association matrices under addition and multiplication but have an order much smaller than that of the association matrices.

It was shown by Connor and Clatworthy (1954) and Bose and Mesner (1959) that if N_d is the incidence matrix of an m-associate PBIB design based on an association scheme $\mathcal{A}$ with m classes, then the eigenvalues of the matrix

$$N_dN_d' = rB_0 + \lambda_1 B_1 + \cdots + \lambda_m B_m \tag{4.3.10}$$

and those of

$$\mathcal{P} = r\mathcal{P}_0 + \lambda_1\mathcal{P}_1 + \cdots + \lambda_m\mathcal{P}_m \tag{4.3.11}$$

are the same. This fact enables one to compute the eigenvalues of the $v \times v$ matrix N_dN_d' in terms of the square matrix $\mathcal{P}$, which is of order $m+1$ only.

Following Bose and Mesner (1959), the eigenvalues θ_i (and their respective multiplicities, α_i) of N_dN_d' where N_d is the incidence matrix of a connected, *two-associate* PBIB design, can be determined as follows: Putting $m=2$ in (4.3.3), we get

$$I_v + B_1 + B_2 = J_v. \tag{4.3.12}$$

From (4.3.2), it is clear that $\mathbf{1}_v$ is an eigenvector of B_i corresponding to an eigenvalue n_i, $i=1,2$. Let $\boldsymbol{x}$ be an eigenvector of B_1 corresponding to an eigenvalue α, say, where $\alpha \neq n_1$. Then, since $\mathbf{1}_v'\boldsymbol{x}=0$, it follows that $\boldsymbol{x}$ is also an eigenvector of B_2 corresponding to an eigenvalue, $-(1+\alpha)=\beta$, say. These facts, together with (4.3.10) imply that $\boldsymbol{x}$ is also an eigenvector of N_dN_d' corresponding to the eigenvalue $r+\lambda_1\alpha+\lambda_2\beta$.

Putting $m=2$ and $s=j=1$ in (4.3.4) and using (4.3.12), one gets

$$B_1^2 = (n_1 - p_{11}^1)I + (p_{11}^1 - p_{11}^2)B_1 + p_{11}^2 J. \tag{4.3.13}$$

Postmultiplying both sides of (4.3.13) by $\boldsymbol{x}$, we get

$$\alpha^2\boldsymbol{x} = (n_1 - p_{11}^1)\boldsymbol{x} + (p_{11}^1 - p_{11}^2)\alpha\boldsymbol{x}. \tag{4.3.14}$$

Since $\boldsymbol{x} \neq \mathbf{0}$, α must satisfy the quadratic equation

$$\alpha^2 - (p_{11}^1 - p_{11}^2)\alpha - (n_1 - p_{11}^1) = 0. \tag{4.3.15}$$

It follows then that B_1 has two distinct eigenvalues other than n_1 and these can be obtained by solving the equation (4.3.15). The eigenvalues of N_dN_d' can now be obtained. Their respective multiplicities can be determined from the relation $\text{tr}(N_dN_d') = vr$ and the fact that for a connected design d, rk is a simple eigenvalue of N_dN_d'.

The eigenvalues of N_dN_d' (and their respective multiplicities) when d is a connected two-associate PBIB design were first obtained by Connor and Clatworthy (1954) following a different approach. These of course can be obtained following the approach outlined above. The explicit expressions for these are given below:

$$\begin{aligned}
\theta_0 &= rk,\ \alpha_0 = 1, \\
\theta_i &= r - \frac{1}{2}\left[(\lambda_1 - \lambda_2)\{-\gamma + (-1)^i\sqrt{\Delta}\} + (\lambda_1 + \lambda_2)\right],\ i = 1, 2 \\
\alpha_i &= \frac{n_1 + n_2}{2} + \frac{(-1)^i}{2\sqrt{\Delta}}\left[(n_1 - n_2) + \gamma(n_1 + n_2)\right],\ i = 1, 2, \quad (4.3.16)
\end{aligned}$$

where

$$\gamma = p_{12}^2 - p_{12}^1,\ \beta = p_{12}^1 + p_{12}^2,\ \Delta = \gamma^2 + 2\beta + 1. \quad (4.3.17)$$

The eigenvalues (and their multiplicities) of N_dN_d' can sometimes be used to prove the nonexistence of a PBIB design in the following way. Since the multiplicities are necessarily integral and involve only the parameters of the parent association scheme, this places a restriction on the parameters of the association scheme. Similarly, since the eigenvalues of N_dN_d' are necessarily nonnegative, it places a restriction on the parameters of the PBIB design.

For a comprehensive and elegant account of association schemes and related combinatorics, a reference may be made to Bailey (2004).

4.4 PBIB Designs with Two Associate Classes

Among the PBIB designs, the ones with two-associate classes are the most important, these being the closest to a balanced design and consequently, these designs have received a great amount of attention. An extensive catalog of two-associate PBIB designs was prepared by Clatworthy (1973), which is an updated and improved version of an earlier catalog by Bose, Clatworthy and Shrikhande (1954). In this section, important results on two-associate PBIB designs are reviewed.

4.4.1 Group-divisible Designs

We first define a group-divisible (GD) association scheme.

Definition 4.4.1 *For integers $m \geq 2$ and $n \geq 2$, consider $v = mn$ treatments, which are arranged in an $m \times n$ array, say B. A group-divisible association scheme on these v treatments is defined as follows: two treatments are first associates if they belong to the same row of B and, second associates, otherwise.*

The rows of the array B have traditionally been called "groups" and hence the name "group-divisible". Though this nomenclature is not very appropriate as the "groups" here have nothing to do with the familiar mathematical notion of groups, we continue to use the same to be consistent with the existing literature. Incidentally, some authors have used the term "groop divisible", possibly to avoid the confusion; however, this terminology has not been accepted widely.

The parameters of the GD association scheme are as follows:

$$\begin{aligned} v &= mn, \;\; n_1 = n-1, \;\; n_2 = n(m-1), \\ P_1 = (p^1_{ij}) &= \begin{bmatrix} n-2 & 0 \\ 0 & n(m-1) \end{bmatrix}, \\ P_2 = (p^2_{ij}) &= \begin{bmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{bmatrix}. \end{aligned} \tag{4.4.1}$$

An incomplete block design is said to be group-divisible if it is based on the GD association scheme. If d is a GD design with parameters $v = mn, b, r, k, \lambda_1, \lambda_2$ and incidence matrix N_d, then the eigenvalues, $\theta_i, i = 0, 1, 2$, of $N_d N_d'$ and their respective multiplicities α_i, following (4.3.16) and (4.3.17), are given by

$$\begin{aligned} \theta_0 &= rk, \;\; \alpha_0 = 1, \\ \theta_1 &= r - \lambda_1, \;\; \alpha_1 = m(n-1), \\ \theta_2 &= rk - v\lambda_2, \;\; \alpha_2 = m-1. \end{aligned} \tag{4.4.2}$$

Clearly, we must have $r \geq \lambda_1$ and $rk \geq v\lambda_2$. Based on this fact, the GD designs have been classified into the following three classes:
(a) Singular, if $r = \lambda_1$;
(b) Semi-regular, if $r > \lambda_1$ and $rk = v\lambda_2$;
(c) Regular, if $r > \lambda_1$ and $rk > v\lambda_2$.

We now discuss some results on the structure and construction of GD designs.

Theorem 4.4.1 *A two-associate PBIB design is group-divisible if and only if either* $p_{12}^1 = 0$ *or* $p_{12}^2 = 0$. *If* $p_{12}^i = 0$, *the treatments in the same group are ith associates,* $i = 1, 2$.

Proof. If d is a GD design on $v = mn$ treatments with m groups of n treatments each, and a pair of treatments belonging to the same group are first associates, then from (4.4.1), we have $p_{12}^1 = 0$. Conversely, for a two-associate PBIB design, suppose $p_{12}^1 = 0$. Then, from (4.2.1), we have $p_{11}^1 = n_1 - 1$. Let x_0 and x_1 be two treatments that are mutually first associates. Let the other first associates of x_0 be $x_2, x_3, \ldots, x_{n_1}$. By virtue of the fact that $p_{11}^1 = n_1 - 1$, x_0 and x_1 have exactly $n_1 - 1$ common first associates and these have to be the treatments $x_2, \ldots, x_{n_1}$. Also, the first associates of x_1 are precisely the treatments $x_0, x_2, x_3, \ldots, x_{n_1}$. It follows that a first associate of x_0 (other than x_1) is also a first associate of x_1. This implies that the v treatments can be partitioned into sets of $(n_1 + 1)$ treatments each, such that a pair of treatments belonging to the same set are first associates while a pair of treatments belonging to different sets are mutually second associates. Thus, the design must be a GD design. In a similar way, one can show that if $p_{12}^2 = 0$, then the design is again a GD design with a pair of treatments belonging to the same group being declared as second associates. □

The following result relates a singular GD design with a BIB design.

Theorem 4.4.2 *The existence of a BIB design with parameters* $v_1, b_1, r_1, k_1, \lambda$ *is equivalent to that of a singular group-divisible design with parameters* $v = nv_1, b = b_1, r = r_1, k = nk_1, \lambda_1 = r_1, \lambda_2 = \lambda, m = v_1, n$.

The next two results provide a lower bound to the number of blocks in semi-regular and regular GD designs.

Theorem 4.4.3 *For a semi-regular group-divisible design with parameters* $v = mn, b, r, k, \lambda_1, \lambda_2$, *the inequality* $b \geq v - m + 1$ *holds. Furthermore, if the semi-regular design is resolvable, then* $b \geq v - m + r$.

Proof. Let d be a semi-regular GD design. Then, for such a design, $rk = v\lambda_2$. If N_d is the incidence matrix of d, then an eigenvalue of $N_d N_d'$ is $rk - v\lambda_2 = 0$ with multiplicity $m - 1$. Hence we have

$$v - (m - 1) = v - m + 1 = Rank(N_d N_d') = Rank(N_d) \leq b. \quad (4.4.3)$$

This proves the first assertion. If the design d is resolvable, N_d consists of r sets of b/r columns each, such that within each set a unity appears once and only once in each row of the set. Adding the first, second, ..., $(b/r-1)$th columns of a set to the (b/r)th column of the same set, we get the column **1**. As there are in all r sets, we have $Rank(N_d) \leq b-(r-1) = b-r+1$. Combining this fact with the one in (4.4.3), we get the required result. □

On similar lines, one can prove the following result.

Theorem 4.4.4 *For a regular group-divisible design, $b \geq v$. Furthermore, if such a design is resolvable, we have the sharper inequality $b \geq v+r-1$.*

We now have the following result in the context of a semi-regular GD design.

Theorem 4.4.5 *For a semi-regular group-divisible design, $m|k$. If in view of this fact, we write $k = \alpha m$, then each block contains α treatments from each group.*

Proof. Consider a semi-regular design with parameters $v, b, r, k, \lambda_1, \lambda_2, m, n$. Let y_{ij} denote the number of treatments from the ith group in the jth block, $1 \leq i \leq m$, $1 \leq j \leq b$. Then, it is easy to see that

$$\sum_{j=1}^{b} y_{ij} = nr, \; \sum_{j=1}^{b} y_{ij}(y_{ij}-1) = n(n-1)\lambda_1. \tag{4.4.4}$$

If $\bar{y}$ denotes the arithmetic mean of the y_{ij} values, then,

$$\bar{y} = nr/b = nk/v = k/m = \alpha, \text{ say.} \tag{4.4.5}$$

Define S.S.$(y_{ij}) = \sum_{j=1}^{b} y_{ij}^2 - b(\bar{y})^2$. Then, using (4.4.4), (4.2.4), the fact that $rk - v\lambda_2 = 0$, and simplifying, we have

$$\begin{aligned} \text{S.S.}(y_{ij}) &= n(n-1)\lambda_1 + nr - bk^2/m^2 \\ &= (n^2\lambda_2 - nrk/m) = 0. \end{aligned} \tag{4.4.6}$$

It follows then $y_{i1} = y_{i2} = \cdots = y_{ib} = \bar{y} = \alpha$. This completes the proof. □

We now take up some major methods of construction of GD designs. By virtue of Theorem 4.4.2, the construction of a singular GD design does not pose any special problem and can be constructed by simply replacing each treatment in a BIB design involving m treatments by a set of n treatments, the sets corresponding to distinct treatments of the BIB design being mutually disjoint. The next result also uses BIB designs for the construction of semi-regular GD designs. Given an incomplete block design d, one can derive another incomplete block design d', called the *dual* of d by interchanging the roles of treatments and blocks in d. We then have the following result.

Theorem 4.4.6 *The dual of an affine resolvable BIB design with parameters $v_1, b_1, r_1, k_1, \lambda$ is a semi-regular group-divisible design with parameters $v = b_1, b = v_1, r = k_1, k = r_1, \lambda_1 = 0, \lambda_2 = k_1^2/v_1, m = r_1, n = b_1/r_1$.*

Example 4.4.1 We illustrate the above theorem via an example. Let d be a BIB design with parameters $v_1 = 9, b_1 = 12, r_1 = 4, k_1 = 3, \lambda = 1$, whose block contents are given in Example 3.2.2. If d' is the dual of this BIB design, then the block contents of d' are as follows, where the treatments are labeled as $0, 1, \ldots, 11$:

$$\begin{array}{ccc}
(0,3,6,9), & (0,4,7,10), & (0,5,8,11), \\
(1,3,7,11), & (1,4,8,9), & (1,5,6,10), \\
(2,3,8,10), & (2,4,6,11), & (2,5,7,9).
\end{array}$$

The design d' can be verified to be a semi-regular GD design with parameters $v = 12, b = 9, r = 3, k = 4, \lambda_1 = 0, \lambda_2 = 1, m = 4, n = 3$.

Semi-regular GD designs with $\lambda_1 = 0$ coexist with a certain family of orthogonal arrays of strength two. For completeness, we first recall the definition of a symmetric orthogonal array.

Definition 4.4.2 *An $M \times N$ matrix A with entries from a finite set of $t \geq 2$ distinct symbols is said to be a (symmetric) orthogonal array of strength $g (2 \leq g < M)$ if in each $g \times N$ submatrix of A, each possible combination of the t symbols appears equally often as a column.*

An orthogonal array will be denoted by $OA(N, M, t, g)$. From Definition 4.4.2, it follows that for an $OA(N, M, t, g)$, $N = \mu t^g$ for some positive integer μ. The integer μ is called the index of the array. Orthogonal arrays were introduced by Rao (1947a) and have been studied extensively in the literature. For comprehensive accounts of orthogonal arrays and their applications, we refer to Hedayat, Sloane and Stufken (1999) and Dey and Mukerjee (1999). We now have the following result due to Bose, Shrikhande and Bhattacharya (1953).

Theorem 4.4.7 *The existence of a semi-regular group-divisible design with parameters $v = mn$, $b = n^2\lambda_2, r = n\lambda_2, k = m, \lambda_1 = 0, \lambda_2, m, n$ is equivalent to that of an orthogonal array $OA(\lambda_2 n^2, m, n, 2)$.*

Proof. Let d be a semi-regular GD design with parameters $v = mn$, $b = n^2\lambda_2, r = n\lambda_2, k = m, \lambda_1 = 0, \lambda_2, m, n$ and for $1 \leq i \leq m$, let the treatments in the ith group of d be

$$\{(i-1)n+0, (i-1)n+1, \ldots, (i-1)n+(n-1)\}.$$

From Theorem 4.4.5, each block of d contains $\alpha = k/m = 1$ treatment from each of the m groups. Form an $m \times b$ array A whose columns are the blocks of d such that the treatments from the ith group form the i row $(1 \leq i \leq m)$. Next, replace the treatment $(i-1)n + j$, of the ith group by j for $j = 0, 1, \ldots, n-1$ to obtain an $m \times n^2\lambda_2$ array B. Then, it can be seen that B is an orthogonal array $OA(n^2\lambda_2, m, n, 2)$. The converse can be proved by tracing back the above steps. □

Example 4.4.2 Consider the semi-regular GD design of Example 4.4.1. For this design, the groups are

$$\begin{array}{ccc} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \end{array}$$

Following the method of construction outlined above, we get the array A as

$$A = \begin{array}{ccccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 3 & 4 & 5 & 3 & 4 & 5 & 3 & 4 & 5 \\ 6 & 7 & 8 & 7 & 8 & 6 & 8 & 6 & 7 \\ 9 & 10 & 11 & 11 & 9 & 10 & 10 & 11 & 9 \end{array}.$$

Replacing the treatment $(i-1)n + j$ of the ith group by j, we get the following array:

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \end{bmatrix}.$$

The array B is easily verified to be an orthogonal array $OA(9, 4, 3, 2)$.

The next construction is also based on a BIB design.

Theorem 4.4.8 *The existence of a BIB design with parameters $v_1, b_1, r_1, k_1, \lambda = 1$ implies that of a group-divisible design with parameters $v = v_1 - 1, b = b_1 - r_1, r = r_1 - 1, k = k_1, \lambda_1 = 0, \lambda_2 = 1, m = r_1, n = k_1 - 1$.*

Proof. Let d_1 be a BIB design with parameters $v_1, b_1, r_1, k_1, \lambda = 1$. Consider a particular treatment α in d_1. Suppose the blocks of d_1 containing α are labeled as $B'_1, B'_2, \ldots, B'_{r_1}$. By deleting these r_1 blocks from d_1, we get a design d with $v = v_1 - 1$ treatments and $b = b_1 - r_1$ blocks. We now show that d is a group-divisible design. From the blocks $\{B'_i\}$, delete α and call the resultant blocks as $B_1, B_2, \ldots, B_{r_1}$. Since $\lambda = 1$, any two blocks of d_1 intersect in at most one treatment. This shows that the blocks $\{B_i,\ 1 \le i \le r_1\}$ are mutually disjoint. The $v = v_1 - 1 = r_1(k_1 - 1)$ treatments form r_1 groups of $k_1 - 1$ treatments each and in fact, the contents of the blocks B_i, $1 \le i \le r_1$, are these groups. Thus, $m = r_1, n = k_1 - 1$. A pair of treatments belong to the same group if and only if they appear together in the same block of d_1 as α. It follows then that $\lambda_1 = 0, \lambda_2 = 1$. The expression for the other parameters of d are obvious. Note that d is a regular GD design if $r_1 > k_1$ and is semi-regular if $r_1 = k_1$. □

A method of construction of GD designs was provided by Dey and Balasubramanian (1991), which is as follows. For $i = 1, 2$, let N_i be a (0,1) matrix of order $v_1 \times b_1$, satisfying

$$\begin{aligned} N_1N'_1 + (t-1)N_2N'_2 &= (r - \lambda_2)I_{v_1} + \lambda_2 J_{v_1} && (4.4.7)\\ N_1N'_2 + N_2N'_1 + (t-2)N_2N'_2 &= (\lambda_1 - \lambda_2)I_{v_1} + \lambda_2 J_{v_1} && (4.4.8)\\ \mathbf{1}'_{v_1}(N_1 + (t-1)N_2) &= k\mathbf{1}'_{b_1}, && (4.4.9)\end{aligned}$$

where $t \ge 2, r \ge 2, \lambda_1 \ge 0, \lambda_2 > 0, k \ge 2$ are integers, $r \ge \max(\lambda_1, \lambda_2)$, such that

$$v_1 r = b_1 k \quad \text{and} \quad r(k-1) = (t-1)\lambda_1 + t(v-1)\lambda_2. \qquad (4.4.10)$$

We then have the following result.

Theorem 4.4.9 *Let N_1, N_2 be matrices satisfying (4.4.7)–(4.4.10). Then,*

$$N = I_t \otimes N_1 + (J_t - I_t) \otimes N_2 \qquad (4.4.11)$$

is the incidence matrix of a group-divisible design with parameters

$$v = tv_1, b = tb_1, r, k, \lambda_1, \lambda_2, m = v_1, n = t. \qquad (4.4.12)$$

Proof. Clearly, N is a (0,1) matrix of order $tv_1 \times tb_1$ and thus, the expressions for v and b in (4.4.12) follow. Let the $v = tv_1$ treatments be partitioned into v_1 equivalence classes each containing t treatments, such that the ith class contains the treatment labels

$$\{i, i+v_1, i+2v_1, \ldots, i+(t-1)v_1\},\ 1 \le i \le v_1. \tag{4.4.13}$$

Define a pair of treatments to be mutually first associates if they belong to the same equivalence class and, second associates otherwise. The proof is completed by using (4.4.7)–(4.4.9) in (4.4.11). □

The above result generalizes an earlier result of Dey (1977), who considered the special case of $t = 2$ and $\lambda_1 = 0$. Several families of GD designs can be constructed via Theorem 4.4.10 and we refer the reader to Dey (1977) and Dey and Balasubramanian (1991) for details on these.

The use of finite geometries has been made for the construction of GD designs and we refer to Sprott (1959) and Bose and Chakravarti (1966) for details on these. Similarly, the method of differences has also been used for the construction of GD designs and we refer to Raghavarao (1971) and Bose Shrikhande and Bhattacharya (1953) for details on these. A large number of GD designs with $2 \le r, k \le 10$ are cataloged by Clatworthy (1973). After the publication of the catalog of Clatworthy (1973), several other miscellaneous methods of construction of GD designs have been reported by Freeman (1976), John and Turner (1977), Seberry (1978), Kageyama and Tanaka (1981), Mohan and Kageyama (1983), Dey and Nigam (1985) and Kageyama (1985).

4.4.2 Triangular Designs

An important class of two-associate PBIB designs is based on a triangular association scheme, which is defined as follows.

Definition 4.4.3 *Let there be $v = m(m-1)/2$ treatments ($m \ge 5$) which are arranged in an $m \times m$ array such that the positions on the principal diagonal are left blank, the $m(m-1)/2$ positions above the principal diagonal are filled up by the v treatments and the positions below the principal diagonal are filled up by the v treatments in such a manner that the resultant arrangement is symmetric about the principal diagonal. On this array, a triangular association scheme with two associate classes is defined as follows: A pair of treatments are first associates if they belong to the same row or the same column of the array and, are second associates, otherwise.*

The parameters of the triangular association scheme are

$$\begin{aligned}
v &= m(m-1)/2,\ n_1 = 2(m-2),\ n_2 = \frac{(m-2)(m-3)}{2}, \\
P_1 &= \begin{bmatrix} m-2 & m-3 \\ m-3 & (m-3)(m-4)/2 \end{bmatrix}, \\
P_2 &= \begin{bmatrix} 4 & 2m-8 \\ 2m-8 & (m-4)(m-5)/2 \end{bmatrix}. \qquad (4.4.14)
\end{aligned}$$

Note that for $m = 4$, a triangular scheme reduces to a group-divisible association scheme and for $m = 3$, there is just one associate class.

Example 4.4.3 Let $m = 5$. The triangular association scheme for $v = 10$ treatments, with treatment labels $0, 1, \ldots, 9$, can be displayed as

$$\begin{array}{ccccc}
* & 0 & 1 & 2 & 3 \\
0 & * & 4 & 5 & 6 \\
1 & 4 & * & 7 & 8 \\
2 & 5 & 7 & * & 9 \\
3 & 6 & 8 & 9 & *
\end{array}$$

The first and second associates of each of the treatments can now be written as in the following table.

Treatment	First associates	Second associates
0	1,2,3,4,5,6	7,8,9
1	0,2,3,4,7,8	5,6,9
2	0,1,3,5,7,9	4,6,8
3	0,1,2,6,8,9	4,5,7
4	0,1,5,6,7,8	2,3,9
5	0,2,4,6,7,9	1,3,8
6	0,3,4,5,8,9	1,2,7
7	1,2,4,5,8,9	0,3,6
8	1,3,4,6,7,9	0,2,5
9	2,3,5,6,7,8	0,1,4

A PBIB design is called a triangular design if it is based on a triangular association scheme. Let d be a triangular design with parameters $v = m(m-1)/2, b, r, k, \lambda_1, \lambda_2$ and incidence matrix N_d. The eigenvalues θ_i of $N_d N_d'$ and their respective multiplicities, α_i are given by

$$\begin{aligned}
\theta_0 &= rk,\ \alpha_0 = 1, \\
\theta_1 &= r + (m-4)\lambda_1 - (m-3)\lambda_2,\ \alpha_1 = m-1, \\
\theta_2 &= r - 2\lambda_1 + \lambda_2,\ \alpha_2 = m(m-3)/2. \qquad (4.4.15)
\end{aligned}$$

Analogous to Theorem 4.4.5, the following result due to Raghavarao (1960a) can be proved.

Theorem 4.4.10 *If in a triangular design, $\theta_1 = 0$, then $m|2k$. In such a case, writing $2k/m = \beta$, every block contains β treatments from each of the m rows of the association scheme.*

We now take up some methods of construction of triangular designs. Consider a triangular association scheme on $v = m(m-1)/2$ treatments. Form blocks of size $k = 2$ each by pairing a treatment label i, $1 \le i \le v$, with each of its first associates.

This process gives us $b = m(m-1)(m-2)/2$ blocks of size two each and the design comprising of these blocks is a triangular design. Another family of designs with block size two can be obtained following the above construction by considering the second associates of each treatment, instead of the first associates. One then has the following result due to Clatworthy (1955).

Theorem 4.4.11 *For an integer $m \ge 5$, there exist two families of triangular designs with $v = m(m-1)/2$ treatments, block size $k = 2$ and other parameters given by (4.4.16) and (4.4.17):*

$$b = \frac{m(m-1)(m-2)}{2}, r = 2(m-2), \lambda_1 = 1, \lambda_2 = 0, \tag{4.4.16}$$

$$b = \frac{m(m-1)(m-2)(m-3)}{8}, r = \frac{(m-2)(m-3)}{2}, \lambda_1 = 0, \lambda_2 = 1. \tag{4.4.17}$$

Example 4.4.4 Let $m = 5$. Then the association scheme is as displayed in Example 4.4.3. By pairing each treatment with its first associates to get blocks, one arrives at the solution of a triangular design with parameters $v = 10, b = 30, r = 6, k = 2, \lambda_1 = 1, \lambda_2 = 0$.

As in the case of GD designs, dualization of a specific BIB design leads to a triangular PBIB design. A result in this direction, obtained by Shrikhande (1960) is given below.

Theorem 4.4.12 *Let d be a BIB design with parameters $v_1 = (m-1)(m-2)/2, b_1 = m(m-1)/2, r_1 = m, k_1 = m-2, \lambda = 2$, where $m \ge 5$. Then, d', the dual of d, is a triangular PBIB design with parameters $v = m(m-1)/2, b = (m-1)(m-2)/2, r = m-2, k = m, \lambda_1 = 1, \lambda_2 = 2$.*

Chang, Liu and Liu (1965) proved the following result.

Theorem 4.4.13 *The existence of a BIB design d_1 with parameters $v_1 = m - 1, b_1, r_1, k_1, \lambda$ implies that of a triangular design with parameters $v = m(m-1)/2, b = mb_1, r = 2r_1, k = k_1, \lambda_1 = \lambda, \lambda_2 = 0$.*

Proof. Let us write the triangular association scheme with $m(m-1)/2$ treatments as an $m \times m$ array, as explained earlier. Then, each row of the array has precisely $m - 1$ treatments. Next, write the solution of the design d_1 with the $m - 1$ treatments in each row of the array. This will generate a total of mb_1 blocks. These blocks form the required triangular design. □

Example 4.4.5 Let $m = 5$. The triangular association scheme can be exhibited as a 5×5 array as in Example 4.4.3. Consider a BIB design with $v_1 = m - 1 = 4$ treatments, say $\theta_1, \ldots, \theta_4$, in blocks of size $k_1 = 2$, whose block contents are

$$(\theta_1, \theta_2); (\theta_1, \theta_3); (\theta_1, \theta_4); (\theta_2, \theta_3); (\theta_2, \theta_4); (\theta_3, \theta_4).$$

Writing these blocks, using the rows of the association scheme, we get a triangular design with parameters $v = 10, b = 30, r = 6, k = 2, \lambda_1 = 1, \lambda_2 = 0$. For instance, the blocks generated from the first row of the association scheme are $(0,1); (0,2); (0,3); (1,2); (1,3); (2,3)$ while those obtained from the second row are $(0,4); (0,5); (0,6); (4,5); (4,6); (5,6)$.

In a triangular association scheme with $m \geq 5$, suppose we form blocks by considering the first associates of a treatment θ $(1 \leq \theta \leq v)$. Then these blocks, $m(m-1)/2$ in number, constitute a triangular design with parameters $v = m(m-1)/2 = b, r = n_1 = 2(m-2) = k, \lambda_1 = p_{11}^1 = m - 2, \lambda_2 = p_{11}^2 = 4$.

Similarly, considering the second associates in place of the first associates of each treatment to form blocks, we get a triangular design with parameters $v = m(m-1)/2 = b, r = n_2 = (m-2)(m-3)/2 = k, \lambda_1 = p_{22}^1 = (m-3)(m-4)/2, \lambda_2 = p_{22}^2 = (m-4)(m-5)/2$.

Cheng, Constantine and Hedayat (1984) gave a graph-theoretic method of construction of triangular designs and some of the above methods of construction can be obtained as special cases of their unified method. For details, the original source might be consulted.

4.4.3 Latin Square Type Designs

The Latin square type PBIB designs with two associate classes are based on a Latin square association scheme, defined below.

Definition 4.4.4 *Suppose there are $v = t^2$ treatments, where $t \geq 3$ is an integer. On these treatments, a Latin square association scheme with $i \geq 2$ constraints (called the L_i association scheme) is defined as follows: Arrange the $v = t^2$ treatments in a $t \times t$ array S and assume that $(i-2)$ mutually orthogonal Latin squares of order t are available. Superimpose each of these squares on S. Two treatments are then defined to be first associates if they occur in the same row or same column of S or in positions occupied by the same letter in any of the Latin squares. They are second associates otherwise.*

The parameters of an L_i association scheme are

$$\begin{aligned}
v &= t^2,\ n_1 = i(t-1),\ n_2 = (t-1)(t-i+1), \\
P_1 &= \begin{bmatrix} (i-1)(i-2)+t-2 & (i-1)(t-i+1) \\ (i-1)(t-i+1) & (t-i)(t-i+1) \end{bmatrix}, \\
P_2 &= \begin{bmatrix} i(i-1) & i(t-i) \\ i(t-i) & (t-i)(t-i-1)+t-2 \end{bmatrix}.
\end{aligned} \tag{4.4.18}$$

A PBIB design based on the Latin square association scheme is called a Latin square type PBIB design. If d is a Latin square type PBIB design with parameters $v = t^2, b, r, k, \lambda_1, \lambda_2$ and incidence matrix N_d. then the eigenvalues (θ_i) of $N_d N_d'$ and their respective multiplicities (α_i) are given by

$$\begin{aligned}
\theta_0 &= rk,\ \alpha_0 = 1, \\
\theta_1 &= r + (t-i)\lambda_1 - (t-i+1)\lambda_2,\ \alpha_1 = i(t-1), \\
\theta_2 &= r - i\lambda_1 + (i-1)\lambda_2,\ \alpha_2 = (t-1)(t-i+1).
\end{aligned} \tag{4.4.19}$$

Analogous to Theorem 4.4.5, the following result due to Raghavarao (1960a) can be proved.

Theorem 4.4.14 *If in a PBIB design based on L_2 association scheme, $\theta_1 = 0$, then $t|k$ and in such a case, each block of the design has k/t treatments from each of the rows (or, columns) of the association scheme.*

We now take up some methods of construction of Latin square type PBIB designs.

Theorem 4.4.15 *If t is a prime or a prime power, then a PBIB design based on L_2 association scheme with parameters*

$$v = t^2, b = t(t-1), r = t-1, k = t, \lambda_1 = 0, \lambda_2 = 1 \tag{4.4.20}$$

can be constructed.

Proof. As before, arrange the $v = t^2$ treatments in a $t \times t$ array S. Since t is a prime or a prime power, a complete set of $t-1$ mutually orthogonal Latin squares of order t exists. Call these Latin squares $A_1, A_2, \ldots, A_{t-1}$. Superimpose $A_1, \ldots, A_{t-1}$ in turn on S and form blocks with those treatments that fall under a particular letter of a Latin square. Thus, from the Latin square A_j, we obtain t blocks. Applying this process to all the Latin squares $\{A_j,\ 1 \le j \le t-1\}$, we get a total of $t(t-1)$ blocks which can be verified to constitute the PBIB design with parameters given by (4.4.20). □

Suppose there exists a BIB design d_1 with parameters $v_1 = t, b_1, r_1, k_1, \lambda$. Then we have the following result.

Theorem 4.4.16 *The existence of a BIB design d_1 implies that of a PBIB design based on the L_2 association scheme with parameters $v = t^2, b = 2b_1, r = 2r_1, k = tk_1, \lambda_1 = r_1 + \lambda, \lambda_2 = 2\lambda$.*

Proof. Consider an L_2 association scheme with t^2 treatments. Replace in d_1, the ith treatment by the ith row of the association scheme, $1 \le i \le t$. This gives a design with t^2 treatments in b_1 blocks of size tk_1 each. Next, replace the ith treatment by the ith column of the association scheme to get another b_1 blocks, which, together with the earlier b_1 blocks give a solution of the required design. □

Again, consider the BIB design d_1 as above. If we write the solution of d_1 with the treatments in the rows and the columns of the L_2 association scheme, we get a design involving t^2 treatments and $2tb_1$ blocks. One can then prove the following result.

Theorem 4.4.17 *The existence of a BIB design with parameters $v_1 = t, b_1, r_1, k_1, \lambda$ implies the existence of a PBIB design based on the L_2 association scheme with parameters $v = t^2, b = 2tb_1, r = 2r_1, k = k_1, \lambda_1 = \lambda, \lambda_2 = 0$.*

The next two results are due to Clatworthy (1967a).

Theorem 4.4.18 *A PBIB design based on an L_2 association scheme with parameters $v = t^2 = b, r = 2t - 1 = k, \lambda_1 = t, \lambda_2 = 2$ can be constructed for every integral $t \geq 3$.*

Proof. Consider an L_2 association scheme with $v = t^2$ treatments. From this scheme, form blocks by putting in the ith block, the treatment i and all its first associates, $1 \leq i \leq t^2$. Then, the design so formed is a PBIB design with parameters as stated in the theorem. □

Theorem 4.4.19 *A PBIB design based on the L_2 association scheme with parameters $v = t^2, b = t^2 - t, r = 2(t-1), k = 2t, \lambda_1 = t, \lambda_2 = 2$ can be constructed for every integral $t \geq 3$.*

Proof. Again, consider an L_2 association scheme with t^2 treatments and form blocks by combining all possible pairs of rows and all possible pairs of columns. The design so obtained is the required PBIB design based on the L_2 association scheme. □

In the catalog of Clatworthy (1973), a large number of PBIB designs based on the Latin square association scheme are tabulated with $v \leq 100$, $2 \leq r, k \leq 10$. For a unified method of construction of PBIB designs based on the L_2 association scheme, see Cheng *et al.* (1984).

4.4.4 PBIB Designs Based on Partial Geometries

Bose (1963) defined a partial geometry (r, k, t) as follows.

Definition 4.4.5 *A partial geometry (r, k, t) is a system of undefined points and lines and an underlying incidence relation satisfying the following conditions:*
(a) any two points are incident with not more than one line;
(b) each point is incident with r lines;
(c) each line is incident with k points;
(d) if a point P is not incident with a line ℓ, then there are exactly $t(\geq 1)$ lines passing through P and intersecting ℓ.

Let us identify the points of a partial geometry (r, k, t) with treatments and define two treatments as first associates if the points corresponding to this pair of treatments are incident with a line of the geometry and, second associates, otherwise. Then, this association rule is an association scheme with two classes and with parameters

$$v = kt^{-1}\{(r-1)(k-1)+t\}, \; n_1 = r(k-1),$$

$$
\begin{aligned}
n_2 &= t^{-1}(r-1)(k-1)(k-t), \\
P_1 &= \begin{bmatrix} (t-1)(r-1)+k-2 & (r-1)(k-t) \\ (r-1)(k-t) & t^{-1}(r-1)(k-t)(k-t-1) \end{bmatrix}, \\
P_2 &= \begin{bmatrix} rt & r(k-t-1) \\ r(k-t-1) & t^{-1}\{(r-1)(k-1)(k-2t)+t(rt-k)\} \end{bmatrix}.
\end{aligned}
\tag{4.4.21}
$$

Treating the lines of a partial geometry (r, k, t) as blocks, one gets a PBIB design with two-associate classes and parameters

$$
v = kt^{-1}\{(r-1)(k-1)+t\}, b = rt^{-1}\{(r-1)(k-1)+t\}, r, k, \lambda_1 = 1, \lambda_2 = 0,
\tag{4.4.22}
$$

where $1 \le t \le r$, $1 \le t \le k$. It can be seen that the partial geometry $(2, m-1, 2)$ is a triangular design with rows of the triangular association scheme with $m(m-1)/2$ treatments as blocks. Some other designs constructed e.g., by Clatworthy (1954), Bose and Clatworthy (1955), Seiden (1961), Raychaudhuri (1962), Shrikhande (1965) and Benson (1966) are now known to be based on partial geometries.

Given a partial geometry (r, k, t) there exists a dual geometry (k, r, t) obtained by interchanging the roles of lines and points in the original geometry. Bose (1963) showed that a necessary condition for the existence of a partial geometry (r, k, t) is that

$$
\frac{rk(r-1)(k-1)}{t(r+k-t-1)}
$$

must be integral. Clatworthy (1973) lists 15 PBIB designs based on partial geometries. The solutions of four more PBIB designs based on partial geometries were given by Dey (1988). For a review of partial geometries and related structures, a reference may be made to Thas (2007).

4.4.5 Cyclic Designs with Two-associate Classes

Though the term *cyclic designs* refers to a wide class of designs, in this subsection we consider only those cyclic designs that are based on a two-class cyclic association scheme. The general cyclic designs will be considered later in this chapter. We begin the discussion by defining a two-class cyclic association scheme.

Definition 4.4.6 *An association scheme with two classes and involving v treatments is called a two-associate cyclic association scheme if*

the first associates of the treatment labeled i are $(i+e_1, i+e_2, \ldots, i+e_{n_1})$ mod v, other treatments being second associates of i, where the e_i's satisfy the following conditions:
(a) the e_i's are all distinct and for $1 \le j \le n_1$, $0 < e_j < v$;
(b) among the $n_1(n_1-1)$ differences $e_i - e_j$ reduced mod v, each of the elements $e_1, \ldots, e_{n_1}$ appears α times and each of the elements $f_1, \ldots, f_{n_2}$ appears β times, where $e_1, \ldots, e_{n_1}$ and $f_1, \ldots, f_{n_2}$ are distinct nonzero elements of the additive group of residue classes mod (v) and $\alpha \neq \beta$.

The parameters of a two-associate cyclic association scheme are v, n_1, n_2 and

$$\begin{aligned} P_1 &= \begin{bmatrix} \alpha & n_1-\alpha-1 \\ n_1-\alpha-1 & n_2-n_1+\alpha+1 \end{bmatrix}, \\ P_2 &= \begin{bmatrix} \beta & n_1-\beta \\ n_1-\beta & n_2-n_1+\beta-1 \end{bmatrix}. \end{aligned} \tag{4.4.23}$$

Next, some construction methods of cyclic designs with two-associate classes are considered. These are due to Clatworthy (1956, 1973).

Suppose $e_1 = 2$, $e_2 = 3$ mod 5 and $n_1 = 2$. Among the differences $e_j - e_{j'}$, $1 \le j \neq j' \le n_1$, the elements $e_1 = 2, e_2 = 3$ do not appear at all (i.e., $\alpha = 0$), while the elements $f_1 = 1, f_2 = 4$ appear $\beta = 1$ times. Thus, we have a cyclic association scheme with parameters $v = 5, n_1 = 2 = n_2, \alpha = 0, \beta = 1$ and it follows that the first associates of treatment label i are $(i+2, i+3)$. The remaining treatments are second associates of i. Designs for $v = 5$, $r \le 10$, based on this cyclic association scheme and their respective parameters are given below, where each initial block is to be developed mod 5:

No.	Parameters	Initial Block(s)
(i)	$b=5, r=2=k, \lambda_1=1, \lambda_2=0.$	$(1,3).$
(ii)	$b=5, r=3=k, \lambda_1=2, \lambda_2=1.$	$(1,2,4).$
(iii)	$b=10, r=6, k=3, \lambda_1=4, \lambda_2=2.$	Repeat Design(ii).
(iv)	$b=15, r=6, k=2, \lambda_1=2, \lambda_2=1.$	$(1,3); (1,3); (1,2).$
(v)	$b=15, r=9, k=3, \lambda_1=5, \lambda_2=4.$	$(1,2,4); (1,2,4)$ $(1,2,5).$
(vi)	$b=25, r=10, k=2, \lambda_1=4, \lambda_2=1.$	$(1,3); (1,3); (1,3);$ $(1,3); (1,2).$

Consider the residue classes mod 13 and let the elements e_i with $n_1 = 6$ be

$$e_1 = 2, e_2 = 5, e_3 = 6, e_4 = 7, e_5 = 8, e_6 = 11.$$

Then, among the 30 differences among these elements, the elements $\{e_i\}$ appear $\alpha = 2$ times each and the elements $f_1 = 1, f_2 = 3, f_3 = 4, f_4 = 9, f_5 = 10, f_6 = 12$ appear $\beta = 3$ times each. Thus, we have a cyclic association scheme with $v = 13, n_1 = 6 = n_2, \alpha = 2, \beta = 3$. Solutions of known cyclic designs with $v = 13$ are given below; each initial block is to be developed mod 13.

No.	Parameters	Initial Block(s)
(*i*)	$b = 13, r = 3 = k, \lambda_1 = 1, \lambda_2 = 0$.	$(1,3,9)$.
(*ii*)	$b = 13, r = 6 = k, \lambda_1 = 3, \lambda_2 = 2$.	$(0,1,2,4,7,9)$.
(*iii*)	$b = 13, r = 7 = k, \lambda_1 = 4, \lambda_2 = 3$.	$(1,2,4,6,7,8,12)$.
(*iv*)	$b = 13, r = 10 = k, \lambda_1 = 8, \lambda_2 = 7$.	$(1,2,3,4,5,7,8,9,10,12)$.
(*v*)	$b = 26, r = 6, k = 3, \lambda_1 = 2, \lambda_2 = 0$.	Repeat Design(*i*).
(*vi*)	$b = 26, r = 8, k = 4, \lambda_1 = 1, \lambda_2 = 3$.	$(1,4,12,13); (1,4,10,13)$.
(*vii*)	$b = 39, r = 6, k = 2, \lambda_1 = 1, \lambda_2 = 0$.	$(1,3); (1,6); (1,7)$.
(*viii*)	$b = 39, r = 9, k = 3, \lambda_1 = 3, \lambda_2 = 0$.	Repeat Design(*i*) thrice.
(*ix*)	$b = 39, r = 9, k = 3, \lambda_1 = 1, \lambda_2 = 2$.	$(0,1,12); (0,3,10); (0,4,9)$,

With $v = 17$ treatments and $n_1 = 8$, if we take the elements $\{e_i\}$ as

$$e_1 = 3, e_2 = 5, e_3 = 6, e_4 = 7, e_5 = 10, e_6 = 11, e_7 = 12, e_8 = 14,$$

then we have a cyclic association scheme with $\alpha = 3, \beta = 4$. Solutions of cyclic designs with at most 10 replications are given below; each initial block is to be developed mod 17.

No.	Parameters	Initial Block(s)
(*i*)	$b = 17, r = 8 = k, \lambda_1 = 4, \lambda_2 = 3$.	$(1,2,4,8,9,13,15,16)$.
(*ii*)	$b = 17, r = 9 = k, \lambda_1 = 5, \lambda_2 = 4$.	$(1,4,6,7,8,11,12,13,15)$.
(*iii*)	$b = 34, r = 8, k = 4, \lambda_1 = 1, \lambda_2 = 2$.	$(0,1,4,5); (1,8,10,16)$.
(*iv*)	$b = 68, r = 8, k = 2, \lambda_1 = 1, \lambda_2 = 0$.	$(1,4); (1,6); (1,7); (1,8)$.

Some other cyclic designs with $v = 29, 37$ have also been reported by Clatworthy (1973) and we refer to the catalog of Clatworthy (1973) for details.

The two-class association schemes and PBIB designs based on them that we have described so far cover all the major types. There are certain other two-class association schemes like the pseudo-triangular, pseudo-Latin square and pseudo-cyclic association schemes that have received attention in the literature. We refer to Shrikhande (1959a), Chang (1960), Hoffman (1960), Seiden (1966) and Mesner (1967) for more details on these.

4.5 PBIB Designs with More Than Two Classes

While the PBIB designs with two-associate classes are the most important among all the PBIB designs, designs with more than two classes have also been studied quite extensively in the literature. In this section, we cover a few aspects of PBIB designs with more than two-associate classes.

4.5.1 Rectangular Designs

These designs were introduced by Vartak (1955). We first define a rectangular association scheme.

Definition 4.5.1 *Let there be $v = t_1 t_2$ treatments, which are arranged in a $t_1 \times t_2$ array, say B. On this array, a rectangular association scheme with three associate classes is defined as follows: A pair of treatments are first associates if they belong to the same row of B, second associates if they belong to the same column of B and, third associates, otherwise.*

The parameters of a rectangular association scheme are

$$
\begin{aligned}
v &= t_1 t_2, n_1 = t_2 - 1, n_2 = t_1 - 1, n_3 = (t_1 - 1)(t_2 - 1), \\
P_1 &= \begin{bmatrix} t_2 - 2 & 0 & 0 \\ 0 & 0 & t_1 - 1 \\ 0 & t_1 - 1 & (t_1 - 1)(t_2 - 2) \end{bmatrix}, \\
P_2 &= \begin{bmatrix} 0 & 0 & t_2 - 1 \\ 0 & t_1 - 2 & 0 \\ t_2 - 1 & 0 & (t_1 - 2)(t_2 - 1) \end{bmatrix}, \\
P_3 &= \begin{bmatrix} 0 & 1 & t_2 - 2 \\ 1 & 0 & t_1 - 2 \\ t_2 - 2 & t_1 - 2 & (t_1 - 2)(t_2 - 2) \end{bmatrix}.
\end{aligned}
\tag{4.5.1}
$$

A PBIB design with three associate classes is called a rectangular design if it is based on the rectangular association scheme. A simple way of constructing a rectangular design is as follows. For $i = 1, 2$, let d_i be a BIB design with parameters $v^{(i)}, b^{(i)}, r^{(i)}, k^{(i)}, \lambda^{(i)}$ and incidence matrix N_{d_i}. Then, $N_d = N_{d_1} \otimes N_{d_2}$ is the incidence matrix of a rectangular design with parameters $v = v^{(1)}v^{(2)}, b = b^{(1)}b^{(2)}, r = r^{(1)}r^{(2)}, k = k^{(1)}k^{(2)}, \lambda_1 = r^{(1)}\lambda^{(2)}, \lambda_2 = r^{(2)}\lambda^{(1)}, \lambda_3 = \lambda^{(1)}\lambda^{(2)}, t_1 = v^{(1)}, t_2 = v^{(2)}$.

4.5.2 Generalized Right-angular Designs

A generalized right-angular PBIB design is a four-associate design based on a generalized right-angular association scheme, defined by Tharthare (1965) as follows.

Definition 4.5.2 *Let there be $v = xyz$ treatments, indexed by a triple, say (x_1, x_2, x_3) where $1 \le x_1 \le y$, $1 \le x_2 \le x$, $1 \le x_3 \le z$. On these v treatments, a generalized right-angular association scheme is defined as follows: For any treatment (x_1, x_2, x_3), the first associates are those that differ in the third position, second associates are those that differ in the second position, irrespective of what is in the third position, third associates are those that have the same second coordinate, a different first coordinate, third position being immaterial. All other treatments are fourth associates of (x_1, x_2, x_3).*

The parameters of this association scheme are

$$
\begin{aligned}
v &= xyz, n_1 = z-1, n_2 = z(x-1), n_3 = z(y-1),\\
n_4 &= z(x-1)(y-1),\\
P_1 &= \begin{bmatrix} z-2 & 0 & 0 & 0 \\ & z(x-1) & 0 & 0 \\ & & z(y-1) & 0 \\ & & & z(x-1)(y-1) \end{bmatrix},\\
P_2 &= \begin{bmatrix} 0 & z-1 & 0 & 0 \\ & z(x-2) & 0 & 0 \\ & & 0 & z(y-1) \\ & & & z(x-2)(y-1) \end{bmatrix},\\
P_3 &= \begin{bmatrix} 0 & 0 & z-1 & 0 \\ & 0 & 0 & z(x-1) \\ & & z(y-2) & 0 \\ & & & z(x-1)(y-2) \end{bmatrix},\\
P_4 &= \begin{bmatrix} 0 & 0 & 0 & z-1 \\ & 0 & z & z(x-2) \\ & & 0 & z(y-2) \\ & & & z(x-2)(y-2) \end{bmatrix},
\end{aligned}
\tag{4.5.2}
$$

where, for convenience, only the upper triangle of the symmetric matrices P_i are displayed above. For $x = 2$, the above association scheme reduces to a right-angular association scheme of Tharthare (1963). For more on these schemes and PBIB designs based on these, we refer to Tharthare (1963, 1965).

4.5.3 Designs Based on Factorial Association Schemes

A factorial association scheme is defined as follows.

Definition 4.5.3 *Let there be $v = m_1 m_2 \dots m_p$ treatments, denoted by $\phi(x_1, x_2, \dots, x_p)$ and indexed by p-tuples $(x_1, \dots, x_p)$, where for $1 \leq i \leq p$, $x_i = 1, 2, \dots, m_i$. On these treatments, a factorial association scheme is defined as follows: a pair of treatments $\phi(x_1, \dots, x_p)$ and $\phi(y_1, \dots, y_p)$ are $(u_1, u_2, \dots, u_p)$th associates if*

$$\{u(x_1 - y_1), u(x_2 - y_2), \dots, u(x_p - y_p)\} = (u_1, u_2, \dots, u_p),$$

where $u(z)$ is a function of z such that $u(z) = 0$, if *$z = 0$ and equals 1, otherwise.*

For this association scheme, the associate classes are represented in binary notation. Clearly, the number of associate classes is $2^p - 1$. This scheme has been studied and used, among others, by Hinkelmann and Kempthorne (1963) and Hinkelmann (1964), who called this scheme as an extended group-divisible (EGD) scheme. Note that for $p = 2$, this scheme reduces to a rectangular association scheme. For $p = 3$ and $m_1 = m_2 = m_3 = m$, such schemes have been called a cubic association scheme by Raghavarao and Chandrasekhararao (1964). For arbitrary $p > 3$, such schemes are known as hypercubic and PBIB designs based on such a scheme are called *hypercubic* designs, which have been studied by Kusumoto (1965) and Chang (1989). A simple method of construction of PBIB designs based on the extended group-divisible association scheme is as follows: for $1 \leq i \leq t$, let d_i be a BIB design with parameters $v_i, b_i, r_i, k_i, \lambda^{(i)}$ and incidence matrix N_{d_i}. Then, the design d with incidence matrix N_d is an extended group-divisible design where N_d is given by

$$N_d = N_{d_1} \otimes N_{d_2} \otimes \cdots \otimes N_{d_t}. \tag{4.5.3}$$

For another family of extended group-divisible designs, see Chang and Hinkelmann (1987).

4.5.4 Designs Based on Group-divisible Family of Schemes

A family of association schemes, called the group-divisible family of association schemes, was introduced by Rao (1966). This scheme is defined as follows.

Definition 4.5.4 *Let there be $v = st$ treatments indexed by a pair (x, y), where $1 \leq x \leq t, 1 \leq y \leq s$. A pair of treatments $(x_1, y_1), (x_2, y_2)$ are ith associates if $x_1 = x_2$ and y_1, y_2 are mutually ith associates according to some given association scheme with u classes on s treatments, $1 \leq i \leq u$; $(x_1, y_1), (x_2, y_2)$ are $(u+1)$th associates if $x_1 \neq x_2$.*

The parameters of this association scheme with $u + 1$ associate classes are

$$\begin{aligned}
v &= st, n_i = n_i^* \ (1 \leq i \leq u),\ n_{u+1} = s(t-1), \\
P_i &= \begin{bmatrix} P_i^* & \mathbf{0} \\ \mathbf{0} & s(t-1) \end{bmatrix}, \ 1 \leq i \leq u, \\
P_{u+1} &= \begin{bmatrix} 0 & 0 & \cdots & 0 & n_1 \\ 0 & 0 & \cdots & 0 & n_2 \\ \vdots & & & & \\ 0 & 0 & \cdots & 0 & n_u \\ n_1 & n_2 & \cdots & n_u & s(t-2) \end{bmatrix},
\end{aligned} \tag{4.5.4}$$

where $P_i^* = (p_{jk}^{i\ *})$ and n_i^* are the parameters of the association scheme with u classes. Clearly, for $u = 1$, this scheme reduces to the two-class group-divisible association scheme. Methods of construction of PBIB designs based on this association scheme are given by Rao (1966) and Dey and Midha (1974). A generalization of the group divisible family of association schemes and designs based on them was considered by Saha and Gauri Shankar (1976) and we refer to the original source for more details.

4.5.5 The m-dimensional Triangular Designs

The two-class triangular association scheme, considered in the previous section has been generalized to a scheme with $m > 2$ classes, called the m-dimensional triangular association scheme (T_m). This scheme is defined as follows.

Definition 4.5.5 *Let there be $v = \binom{t}{m}$ treatments denoted by $\phi(x_1, x_2, \ldots, x_m)$ and indexed by the combinations of m integers $(x_1, x_2, \ldots, x_m)$ out of the t integers $1, 2, \ldots, t$. A pair of treatments $\phi(x_1, \ldots, x_m)$ and $\phi(y_1, \ldots, y_m)$ are ith associates if their indices have precisely $m - i$ integers in common $(1 \leq i \leq m)$.*

It is easy to see that for $m = 2$, this scheme reduces to the two associate triangular scheme considered earlier. The parameters of the T_m association scheme are given by

$$\begin{aligned} v &= \binom{t}{m}, n_i = \binom{m}{i}\binom{t-m}{i}, \quad 1 \leq i \leq m, \\ p^i_{jk} &= \sum_{u=0}^{m-i} \binom{m-i}{u}\binom{i}{m-j-u}\binom{i}{m-k-u} \times \\ & \quad \binom{t-m-i}{j+k+u-m}. \end{aligned} \tag{4.5.5}$$

The special case of this association scheme for $m = 3$ was studied by Kusumoto (1965) and John (1966). The T_m scheme for arbitrary m was considered by Ogasawara (1965). Methods of construction of PBIB designs based on the T_m scheme can be found in the above mentioned references and in Saha (1973).

4.5.6 Kronecker Product Designs

It has been observed earlier in this section that some PBIB designs can be obtained by taking the Kronecker (or, tensor) product of incidence matrices of two known designs. In this subsection, we first introduce a Kronecker product (or simply, product) association scheme and then show how designs based on this scheme can be constructed.

For $t = 1, 2$, let $\mathcal{A}_t$ be an association scheme with parameters

$$v^{(t)}, n_i^{(t)}, p_{js}^{(t)i}, \; i, j, s = 1, 2, \ldots, m, \text{ if } t = 1,$$

$$i, j, s = 1, 2, \ldots, n, \text{ if } t = 2.$$

The product association scheme $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is defined as follows:

$\mathcal{A}$ has $v = v^{(1)}v^{(2)}$ treatments indexed by a pair (α, β), $1 \leq \alpha \leq v^{(1)}, 1 \leq \beta \leq v^{(2)}$. A pair of treatments (α, β) and (γ, δ) are mutually (u_1, u_2)th associates if α is the u_1th associate of γ in $\mathcal{A}_1$ and β is the

u_2th associate of δ in $\mathcal{A}_2$, $1 \leq u_1 \leq m$, $1 \leq u_2 \leq n$. Every treatment is $(0,0)$th associate of itself and of no other treatment.

Clearly, the number of associate classes in $\mathcal{A}$, excluding the$(0,0)$th class, is $m + n + mn$. The other parameters of $\mathcal{A}$ can be easily derived. The same idea can be extended easily to obtain a product association scheme based on the product of more than two association schemes.

We may regard a BIB design as a special case of a PBIB design with just one associate class. The corresponding association scheme can be defined as follows: every treatment is the first associate of every other treatment and is the zeroth associate of itself. For convenience, such an association scheme may be referred to as the BIB association scheme. It can then be verified that the product association scheme of t BIB association schemes with $v_1, v_2, \ldots, v_t$ treatments respectively, is an extended group-divisible association scheme with $2^t - 1$ associate classes. If $v_1 = v_2 = \cdots = v_t$, the scheme reduces to a hypercubic association scheme.

Finally, it can be seen easily that if N_1 is the incidence matrix of a PBIB design with m associate classes based on an association scheme $\mathcal{A}_1$ and N_2, that of a PBIB design with n associate classes based on an association scheme $\mathcal{A}_2$, then $N = N_1 \otimes N_2$ is the incidence matrix of a PBIB design based on the product association scheme $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$.

In some situations, the number of associate classes in a product association scheme can be reduced. For an excellent account of the conditions under which the number of classes in a product association scheme (and, other association schemes) can be reduced, see Kageyama (1974).

Remark 4.5.1 The above are some of the major association schemes and related PBIB designs with three or more associate classes. There are several other PBIB designs based on higher associate class schemes; see for example, Roy (1953), Raghavarao (1960b), Adhikary (1967) and Saha, Kulshreshtha and Dey (1973). However, we do not elaborate on these here as these are beyond the scope of this book. For several other results on PBIB designs, including those on existence, see Raghavarao (1971) and Dey (1986).

4.6 Analysis of PBIB Designs

The analysis of a PBIB design with $m \geq 2$ associate classes can be derived from the general results outlined in Chapter 2. In what follows, however, we briefly sketch the essential steps of analysis of a PBIB design with $m \geq 2$ associate classes.

Let d be a connected m-associate PBIB design with parameters $v, b, r, k, \lambda_i, n_i, p^i_{js}$, $i, j, s = 1, 2, \ldots m$, and let as before, $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_v)'$, where τ_i is the effect of the ith treatment, $1 \leq i \leq v$. Under the intra-block model, the reduced normal equations for estimating linear combinations of treatment effects are given by

$$kC_d\boldsymbol{\tau} = k\boldsymbol{Q}. \tag{4.6.1}$$

In view of (4.3.10), kC_d can be expressed as

$$kC_d = r(k-1)B_0 - \sum_{t=1}^{m} \lambda_t B_t. \tag{4.6.2}$$

Hence, one can write (4.6.1) as

$$-\sum_{t=0}^{m} \lambda_t B_t \boldsymbol{\tau} = k\boldsymbol{Q}, \tag{4.6.3}$$

where $\lambda_0 = -r(k-1)$.

For $0 \leq j \leq m$, let $S_j(i)$ denote the set of treatments which are jth associates of treatment i. Recall that $S_0(i) = \{i\}$. Define

$$T_j(i) = \sum_{l \in S_j(i)} \tau_l \ \text{ and } \ T_j(Q_i) = \sum_{l \in S_j(i)} Q_l. \tag{4.6.4}$$

Clearly then,

$$T_j(i) = B_j\boldsymbol{\tau} \ \text{ and } \ T_j(Q_i) = B_j\boldsymbol{Q}. \tag{4.6.5}$$

Premultiplying both sides of (4.6.3) by B_i and using (4.4.5), we get

$$-\sum_{t=0}^{m} \lambda_t \sum_{s=0}^{m} p^s_{it} B_s \boldsymbol{\tau} = kB_i\boldsymbol{Q},$$

which can be written as

$$-\sum_{s=0}^{m} \left(\sum_{t=0}^{m} p^s_{it} \lambda_t \right) B_s \boldsymbol{\tau} = kB_i\boldsymbol{Q}. \tag{4.6.6}$$

Set

$$\begin{aligned} a_{is} &= -\sum_{t=0}^{m} p^s_{it}\lambda_t, \\ x_s &= T_s(\tau_h), \\ b_i &= kT_i(Q_h). \end{aligned} \tag{4.6.7}$$

Then, in view of (4.6.5), the system of equations (4.6.6) can be written as a system of $m+1$ equations

$$A\boldsymbol{x} = \boldsymbol{b}, \tag{4.6.8}$$

where $A = (a_{is}), 0 \le i, s \le m, \boldsymbol{x} = (x_0, x_1, \ldots, x_m)'$ and $\boldsymbol{b} = (b_0, b_1, \ldots, b_m)'$ It can be checked that

$$\sum_{i=0}^{m} a_{is} = 0, \ \sum_{s=0}^{m} x_s = 0 \ \text{ and } \ \sum_{i=0}^{m} b_i = 0.$$

In view of this, we may ignore the 0th equation of (4.6.8) and express x_0 as $-\sum_{s=1}^{m} x_s$ in the other equations to get a system of m equations

$$U\boldsymbol{x}^* = \boldsymbol{b}^*, \tag{4.6.9}$$

where $U = (u_{ij}), 1 \le i, j \le m$, $u_{ij} = a_{ij} - a_{i0}, 1 \le j \le m$, $\boldsymbol{x}^* = (x_1, \ldots, x_m)'$ and $\boldsymbol{b}^* = (b_1, \ldots, b_m)'$. It may be verified that U is non-singular and thus, a solution of the normal equations, $\hat{\tau}_h$, can be obtained by noting that the solution of (4.6.9) is $\boldsymbol{x}^* = U^{-1}\boldsymbol{b}^*$.

For $m = 2$, one can follow the above steps to get a solution of the normal equations explicitly as

$$\hat{\tau}_i = \frac{k(B_2 Q_i - A_2 T_1(Q_i))}{A_1 B_2 - A_2 B_1}, \ 1 \le i \le v, \tag{4.6.10}$$

where

$$\begin{aligned} A_1 &= r(k-1) + \lambda_2, \\ A_2 &= \lambda_2 - \lambda_1, \\ B_1 &= A_2 p_{12}^2, \\ B_2 &= A_1 + (\lambda_2 - \lambda_1)(p_{11}^1 - p_{11}^2). \end{aligned} \tag{4.6.11}$$

The variance of the best linear unbiased estimator of an elementary treatment contrast is given by

$$\begin{aligned} \operatorname{Var}(\hat{\tau}_i - \hat{\tau}_j) &= \frac{2k(A_2 + B_2)\sigma^2}{A_1 B_2 - A_2 B_1}, \\ &\quad \text{if } i \text{ and } j \text{ are first associates}, && (4.6.12) \\ &= \frac{2kB_2\sigma^2}{A_1 B_2 - A_2 B_1}, \\ &\quad \text{if } i \text{ and } j \text{ are second associates}. && (4.6.13) \end{aligned}$$

Next, we briefly take up the issue of recovery of inter-block information in the context of an m-associate PBIB design d. Recall equation (2.4.39) of Chapter 2. Since d is equireplicate, we have here

$$C_{d_M} = (\omega_1 - \omega_2)C_{d_F} + r\omega_2(I_v - v^{-1}J_v). \tag{4.6.14}$$

Now, following the steps of analysis under the intra-block model, we find that $\boldsymbol{x}^* = (x_1, \ldots, x_m)'$ can be obtained by solving the following system of m equations

$$U_M\boldsymbol{x}^* = \boldsymbol{b}_M, \tag{4.6.15}$$

where $U_M = (\omega_1 - \omega_2)U + rk\omega_2 I_v$ and $\boldsymbol{b}_M = (b_{1M}, \ldots, b_{mM})'$ with $b_{j_M} = kT_j(Q_{i_M})$ and Q_{i_M} is the ith component of $\boldsymbol{Q}_M$ given in (2.4.39). The analysis can now be completed.

4.7 Lattice Designs

Lattice designs, also called quasi-factorial designs, form an important class of incomplete block designs. These designs were introduced by Yates (1936b). In the two classes of lattice design that we define and study in this section, the treatments can be identified with the treatment combinations of a symmetrical factorial experiment and using this factorial structure, one can obtain resolvable incomplete block designs by using different patterns of confounding of factorial effects. The nomenclature *quasi-factorial* is justified in view of this analogy.

We first introduce *square lattice* designs. Let there be $v = t^2$ treatments, where $t \geq 2$ is an integer. As in the context of a Latin square association scheme, let us arrange the v treatments in a $t \times t$ array S. Assume furthermore that there exist $i - 2 \leq t - 1$ mutually orthogonal Latin squares of order t and call these squares $A_1, A_2, \ldots, A_{i-2}$. The blocks of an i-ple square lattice design are then obtained as follows:
(i) treat the rows of S as blocks to get t blocks, each of size $k = t$;
(ii) treat the columns of S as blocks to generate another t blocks;
(iii) superimpose the Latin squares $A_1, \ldots, A_{i-2}$ in turn on S and form blocks by putting all those treatments in a block which fall under a particular letter of a Latin square. For each of the Latin squares, we thus get t blocks, and considering all the $i - 2$ Latin squares, one gets a total of $t(i - 2)$ blocks each of size $k = t$.

The collection of $t + t + t(i - 2) = it$ blocks constitute an i-ple square lattice design. From this construction procedure, it is clear that an i-ple square lattice design is a two-associate PBIB design based on the

Latin square association scheme with i constraints (or, an L_i association scheme, $i \geq 2$) with parameters $v = t^2, b = it, r = i, k = t, \lambda_1 = 1, \lambda_2 = 0$. When $i = 2$, we get a simple square lattice design. If $i = t + 1$ (i.e., when a complete set of mutually orthogonal Latin squares is available), a square lattice design reduces to a BIB design (called a balanced lattice design) with parameters $v = t^2, b = t(t+1), r = t+1, k = t, \lambda = 1$; see Remark 3.4.1. Recall that this family of BIB designs was also constructed earlier using finite Euclidean geometry in Chapter 3 (*vide* Theorem 3.4.2).

We may identify the $v = t^2$ treatments of a square lattice design with the treatment combinations of a symmetrical factorial experiment involving two factors, say F_1 and F_2, each at t levels. One can then obtain a square lattice design by obtaining blocks using different confounding patterns in each replicate. For instance, let $t = 2$. The $v = 4$ treatment combinations of a 2^2 factorial experiment can then be written as $\{00, 01, 10, 11\}$, where 0 and 1 denote the two levels of each of the factors. Suppose we choose to confound the main effect of F_1 in the first replicate and the main effect of F_2 in the second replicate. Then, the blocks of the design are as displayed below:

Replicate	Confounded effect	Block 1	Block 2
I	F_1	(00,01)	(10,11)
II	F_2	(00,10)	(01,11)

Relabeling the treatment combinations according to the scheme

$$00 \to 1, 01 \to 2, 10 \to 3, 11 \to 4,$$

we get the following design:

Replicate	Block 1	Block 2
I	(1,2)	(3,4)
II	(1,3)	(2,4)

This design is easily recognized as a simple square lattice design, which could also have been constructed following the construction method described in the beginning of this section.

A *cubic* or three-dimensional lattice design is an incomplete block design involving $v = t^3$ treatments in blocks of size $k = t$. Consider a cube of side t and arrange the treatments on this cube such that the treatment in the ith row, jth column and mth layer is (x_i, y_j, z_m). A

three-dimensional lattice has at least three replicates involving t^2 blocks in each replicate and the blocks of these replicates are as follows:
Replicate 1: $\{(x_i, y_j, z_1), (x_i, y_j, z_2), \cdots, (x_i, y_j, z_t)\}$, $1 \le i, j \le t$.
Replicate 2: $\{(x_i, y_1, z_m), (x_i, y_2, z_m), \cdots, (x_i, y_t, z_m)\}$, $1 \le i, m \le t$.
Replicate 3: $\{(x_1, y_j, z_m), (x_2, y_j, z_m), \cdots, (x_t, y_j, z_m)\}$, $1 \le j, m \le t$.

It can be shown that a cubic lattice design as described above is a three-associate PBIB design based on a cubic association scheme with parameters $v = t^3, b = 3t^2, r = 3, k = t, \lambda_1 = 1, \lambda_2 = 0 = \lambda_3$.

Finally, we describe a *rectangular lattice* design involving $v = t(t-1)$ treatments. These treatments are indexed by a pair (α, β), where $\alpha \ne \beta$. A *simple* rectangular lattice design has $b = 2t$ blocks. One can obtain t blocks by putting all those treatments in a block which have the same first coordinate. Similarly, another t blocks can be obtained by putting all those treatments in a block that have the same second coordinate. Nair (1951) showed that a simple rectangular lattice design is indeed a PBIB design with four associate classes and parameters

$$\begin{aligned}
v &= t(t-1), b = 2t, r = 2, k = t-1, \lambda_1 = 1, \lambda_2 = 0 = \lambda_3 = \lambda_4, \\
n_1 &= 2(t-2), n_2 = (t-2)(t-3), n_3 = 2(t-2), n_4 = 1, \\
P_1 &= \begin{bmatrix} t-3 & t-3 & 1 & 0 \\ & (t-3)(t-4) & t-3 & 0 \\ & & t-3 & 1 \\ & & & 0 \end{bmatrix}, \\
P_2 &= \begin{bmatrix} 2 & 2(t-4) & 2 & 0 \\ & (t-4)(t-5) & 2(t-4) & 1 \\ & & 2 & 0 \\ & & & 0 \end{bmatrix}, \\
P_3 &= \begin{bmatrix} 1 & t-3 & t-3 & 1 \\ & (t-3)(t-4) & t-3 & 0 \\ & & 1 & 0 \\ & & & 0 \end{bmatrix}, \\
P_4 &= \begin{bmatrix} 0 & 0 & 2(t-2) & 0 \\ & (t-2)(t-3) & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix}.
\end{aligned} \tag{4.7.1}$$

While the lattice designs described so far in this section are PBIB designs with two or more associate classes, not all lattice designs are PBIB designs. For more details on such lattice designs, see Nair (1951) and

Harshbarger (1951). See also Rao (1956) for another class of quasi-factorial designs.

It has been shown by Patterson and Williams (1976) that the efficiency factor of a square lattice design with $v = t^2$ treatments and s replicates is given by

$$E = \frac{(v-1)(s-1)}{(v-1)(s-1) + s(t-1)}. \tag{4.7.2}$$

For a proof of this result and more details, we refer to the original source.

4.8 Cyclic Designs

While discussing the two-associate cyclic PBIB designs (Section 4.4.5), we had remarked that the term "cyclic designs" is used in a broader context. In this section, we briefly study a general cyclic design. For a more comprehensive account of cyclic designs including generalizations and applications, we refer to John (1987) and John and Williams (1995).

In Chapter 3, we have described the method of developing initial blocks for constructing BIB designs. The same method can be applied to a set of initial blocks even when these initial blocks do not have the property of symmetrically repeated differences. Designs obtained by developing initial blocks are known by the generic name *cyclic* designs. When the chosen initial blocks do not give rise to symmetrically repeated differences, the cyclic design obtained by developing is not a BIB design, but it may have some desirable properties. Clearly, cyclic designs are easy to construct and very flexible, making these extremely useful in practice. These designs can be analyzed easily, as we shall see presently. A systematic study of cyclic designs in blocks of size two (also called cyclic *paired comparison* designs) was initiated by Kempthorne (1953) with subsequent contributions from Zoellner and Kempthorne (1954), McKeon (1960) and David (1963a,b, 1965). Cyclic designs with arbitrary block sizes have been studied quite extensively; see e.g., Wolock (1964), David and Wolock (1965), Clatworthy (1967b) and John (1966, 1969, 1973).

A cyclic design involving v treatments, block size k and replication number r will be denoted by $C(v, k, r)$. The treatments will invariably be labeled as the elements of residue classes modulo v, i.e., as $0, 1, \ldots, v-1$. If an initial block has treatments $(j_1, j_2, \ldots, j_k)$, then the other blocks are obtained by adding a nonzero element from the set $\{0, 1, \ldots, v-1\}$ and reducing the elements so obtained mod v. For example, let there be two initial blocks, $(1,2,4)$ and $(1,3,7)$ for generating a cyclic design $C(9,3,6)$. Then the full design is obtained by developing these initial blocks mod 9, which is shown below:

$$(1,2,4); (2,3,5); (3,4,6); (4,5,7); (5,6,8); (0,6,7); (1,7,8);$$

$$(0,2,8); (0,1,3);$$

$$(1,3,7); (2,4,8); (0,3,5); (1,4,6); (2,5,7); (3,6,8); (0,4,7);$$

$$(1,5,8); (0,2,6).$$

For a given triple (v, k, r), there will in general be several choices for a design $C(v, k, r)$. In such a scenario, one should choose that design for which the average efficiency factor is a maximum over the class of all designs with the same parameters. A catalog of designs prepared by John, Wolock and David (1972) contains a very large number of most efficient cyclic designs.

The intra-block analysis of cyclic designs can be accomplished by following the theory presented in Chapter 2. Some features special to cyclic designs are described in the following. Under the standard model (2.2.1), it can be seen that the C-matrix of a cyclic design is circulant. Recall that a square matrix of the form

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & & & & \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}$$

is called circulant, i.e., for a circulant matrix, the tth row $(2 \leq t \leq n)$ is obtained from the $(t-1)$st row by moving each element of the $(t-1)$st row one column to the right and placing the element in the last column of the $(t-1)$st row in the first column of the tth row. In view of this, it suffices to display the entries in the first row to describe a circulant matrix.

One can obtain a circulant generalized inverse of the C-matrix following Kempthorne (1953). Suppose $A = (a_{ij})$ is a circulant generalized inverse of the C-matrix of a cyclic design. Since A is circulant, it suffices to specify only the first row of A. The elements of the first row of A are given by

$$\begin{aligned} a_{11} &= v^{-1}\sum_{u=2}^{v}\lambda_u^{-1}, \\ a_{1j} &= v^{-1}\sum_{u=2}^{v}\cos\{(u-1)(j-1)\phi\}\lambda_u^{-1},\ 2 \le j \le v, \end{aligned} \qquad (4.8.1)$$

where

$$\phi = 2\pi/v \ \text{ and } \ \lambda_u = c_{11} + \sum_{u=2}^{v} c_{1u}\cos\{(u-1)(j-1)\phi\} \qquad (4.8.2)$$

and $c_{11}, c_{12}, \ldots, c_{1v}$ are the entries in the first row of the C-matrix, which is circulant. The quantities λ_u are symmetric with λ_{v-u+2} and the elements c_{1j} are given by

$$\begin{aligned} c_{11} &= rk^{-1}(k-1), \\ c_{12} &= c_{1v} = -k^{-1}\mu_{12}, \\ c_{13} &= c_{1,v-1} = -k^{-1}\mu_{13}, \\ &\vdots \\ c_{1,\frac{v+1}{2}} &= c_{1,\frac{v+3}{2}} = -k^{-1}\mu_{1,\frac{v+1}{2}},\ \text{if } v \text{ is odd}, \\ c_{1,\frac{v+2}{2}} &= -k^{-1}\mu_{1,\frac{v+2}{2}},\ \text{if } v \text{ is even}, \end{aligned} \qquad (4.8.3)$$

where, for $1 \le i, j \le v, i \ne j$, μ_{ij} is the number of times treatments i and j appear together in a block of the design. The variance of the BLUE of an elementary treatment contrast under a cyclic design is given by

$$\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_j) = 2\sigma^2(a_{11} - a_{1t}), \qquad (4.8.4)$$

where $t = j - i + 1,\ j > i$. It can then be shown that the average variance of the BLUEs of all elementary treatment contrasts is

$$\text{Average Variance} = \frac{2\sigma^2 a_{11} v}{v-1}. \qquad (4.8.5)$$

It follows then that the efficiency-factor of a cyclic design is given by

$$E = \frac{v-1}{vra_{11}}. \qquad (4.8.6)$$

Following the development in Chapter 2 (Section 2.4) and the notations used therein, one can show that combined intra-inter-block estimate of τ under a cyclic design d with incidence matrix N_d is

$$\hat{\tau} = (\omega_1 rI - k^{-1}(\omega_1 - \omega_2)N_d N_d')^{-1}(\omega_1 \boldsymbol{T} - k^{-1}(\omega_1 - \omega_2)N_d \boldsymbol{B}), \quad (4.8.7)$$

where, as in Chapter 2 (equation (2.4.23)), $\omega_1 = \sigma^{-2}$ and $\omega_2 = (\sigma^2 + k\sigma_b^2)^{-1}$. The matrix

$$V_d = \omega_1 rI - k^{-1}(\omega_1 - \omega_2)N_d N_d' \quad (4.8.8)$$

for the cyclic design d is also circulant and invertible. Suppose the elements in the first row of V_d^{-1} be $v_{11}, \ldots, v_{1v}$. Then, it can be shown that

$$\begin{aligned} v_{11} &= v^{-1}\sum_{u=2}^{v} \gamma_u^{-1}, \\ v_{1j} &= v^{-1}\sum_{u=2}^{v} \cos\{(u-1)(j-1)\phi\}\gamma_u^{-1}, \; 2 \le j \le v, \end{aligned} \quad (4.8.9)$$

where, for $2 \le u \le v$, $\gamma_u = (\omega_1 - \omega_2)\lambda_u + r\omega_2$ and λ_u and ϕ are as in (4.8.2).

John (1973) extended cyclic designs to generalized cyclic (GC) designs, in which there are $v = s_1 s_2 \ldots s_n$ treatments arranged in b blocks of size k each. A treatment is indexed by an n-tuple $(e_1, e_2, \ldots, e_n)$, where for $1 \le i \le n$, $e_i = 0, 1, \ldots, s_i - 1$. A GC design is obtained by developing one or more initial blocks, the jth block developed from an initial block being obtained by adding the jth treatment to each of the contents of the initial block, where addition means entry-wise addition, i.e., $(e_1, e_2, \ldots, e_n) + (f_1, f_2, \ldots, f_n) = (e_1 + f_1, e_2 + f_2, \ldots, e_n + f_n)$, the elements being reduced mod $(s_1, s_2, \ldots, s_n)$. For example, let $v = 16, s_1 = s_2 = s_3 = s_4 = 2$ and suppose $k = 6$. Consider an initial block

$$(0000, 0001, 1000, 1111, 0101, 0011).$$

Then, developing this block mod (2,2,2,2), we get a generalized cyclic design with $r = 6$. In fact, this is a BIB design with parameters $v = 16 = b, r = 6 = k, \lambda = 2$.

Another type of generalized cyclic design has been considered by Jarrett and Hall (1978). In general, these designs are with unequal block sizes and unequal replications. For details, we refer to Jarrett and Hall (1978) and John (1987). For an excellent review of generalized cyclic designs and their use in factorial experiments, see Dean (1990).

4.9 Linked Block Designs

In this section, we briefly study a class of incomplete block designs, called linked block designs, introduced by Youden (1951). These designs are defined below.

Definition 4.9.1 *An incomplete block design is called a linked block design if each pair of blocks intersects in a constant number, say μ, of treatments.*

It is easy to see that if d is a BIB design with parameters v, b, r, k, λ, then its dual design d' is a linked block design with the parameter $\mu = \lambda$. Roy and Laha (1956) classified linked block designs into the following three categories: (a) symmetric BIB designs, (b) PBIB designs and, (c) irregular designs, that are not covered by the types (a) and (b). The designs of type (a) do not offer any special features different from those of a BIB design and are thus of little interest as linked block designs. It is of interest though to examine conditions under which a linked block design is a PBIB design. To that end, we have the following result.

Theorem 4.9.1 *Let d be a PBIB design with m associate classes and parameters $v, b, r, k, \lambda_i, n_i, p^i_{js}$, $i, j, s = 1, 2, \ldots, m$, and suppose N_d is the incidence matrix of d. Then d is a linked block design if and only if $N_d N_d'$ has only two nonzero eigenvalues, $\theta_0 = rk$ and θ_1 with respective multiplicities $\alpha_0 = 1, \alpha_1 = b - 1$.*

Proof. If d is a linked block design with (constant) block intersection number μ, then we have

$$N_d' N_d = (k - \mu) I_b + \mu J_b.$$

It follows then that the eigenvalues of $N_d' N_d$ are $\theta_0 = rk$ and $\theta_1 = k - \mu$ with respective multiplicities $\alpha_0 = 1, \alpha_1 = b - 1$. Since the nonzero eigenvalues of $N_d N_d'$ and those of $N_d' N_d$ are the same, including the multiplicities, $N_d N_d'$ has only two nonzero eigenvalues, rk with multiplicity 1 and $k - \mu$ with multiplicity $b - 1$. Conversely, suppose d is a PBIB design with block size k and replication r such that $N_d N_d'$ has only two nonzero eigenvalues, say $\theta_0 = rk$ and θ_1, with respective multiplicities 1 and $b - 1$. Then $N_d' N_d$, which is a symmetric matrix of order b, has only two eigenvalues, θ_0 and θ_1, the latter having multiplicity $b - 1$. Also, since $N_d' N_d \mathbf{1}_b = rk\mathbf{1}_b = \theta_0 \mathbf{1}_b$, $\mathbf{1}_b$ is an eigenvector corresponding to θ_0. Thus, $N_d' N_d$ is completely symmetric (recall the result given in the proof of Corollary 2.3.1) and is given by $N_d'N_d = (k-\theta_1)I_b + \theta_1 J_b$ as, the diagonal elements of $N_d'N_d$ are each equal to k. □

As an application of the above theorem, consider a singular GD design with $v = mn$ treatments (i.e., there are m groups of n treatments each). From (4.4.2), it follows then that such a design is linked block if and only if $b = m$. Similarly, a semi-regular GD design is linked block if and only if $b = v - m + 1$. Similar results can be obtained for two-associate triangular designs and designs based on the Latin square association scheme.

4.10 C-Designs

In this section, we consider a class of incomplete block designs possessing a certain property, called C-property. This property was introduced by Caliński (1971) and subsequently studied in greater detail by Saha (1976), Puri and Nigam (1977) and Ceranka and Kozlowska (1983). Saha (1976) called the designs having C-property as C-designs while, the same class of designs was termed simple partially efficiency-balanced designs by Puri and Nigam (1977). These designs, though not always balanced, have a simple analysis and this fact seems to be the main motivation for studying these designs.

Consider a *connected* block design with v treatments, b blocks and incidence matrix N_d. Recall the definition of the symmetric matrix A_d of order v from Chapter 2 (equation (2.3.9)). Suppose A_d has only two distinct nonzero eigenvalues, $\epsilon \in (0,1)$ and 1 with respective multiplicities $m, 1 \le m \le v-1$ and $v-1-m$. This in turn means that the design d has only two distinct canonical efficiency factors, ϵ and unity. A connected block design will be called a C-design if it has only two distinct canonical efficiency factors, $\epsilon \in (0,1)$ and unity. It is easily seen that the C-matrix of a C-design d is

$$
\begin{aligned}
C_d &= R_d^{\frac{1}{2}} \left(\epsilon \sum_{i=1}^{m} \xi_{di}\xi_{di}' + \sum_{i=m+1}^{v-1} \xi_{di}\xi_{di}' \right) R_d^{\frac{1}{2}} \\
&= R_d^{\frac{1}{2}} \left(\epsilon \sum_{i=1}^{v-1} \xi_{di}\xi_{di}' + (1-\epsilon) \sum_{i=m+1}^{v-1} \xi_{di}\xi_{di}' \right) R_d^{\frac{1}{2}} \\
&= R_d^{\frac{1}{2}} \left(\epsilon(I_v - \rho_d\rho_d'/n) + (1-\epsilon) \sum_{i=m+1}^{v-1} \xi_{di}\xi_{di}' \right) R_d^{\frac{1}{2}}
\end{aligned}
$$

$$= \epsilon R_d - \epsilon \mathbf{r}_d \mathbf{r}'_d/n + (1-\epsilon) R_d^{\frac{1}{2}} \sum_{i=m+1}^{v-1} \boldsymbol{\xi}_{di} \boldsymbol{\xi}'_{di} R_d^{\frac{1}{2}}, \tag{4.10.1}$$

where, as before, $R_d = \mathrm{diag}(r_{d1}, \ldots, r_{dv})$, n is the total number of experimental units in d and without loss of generality, the positive eigenvalues of A_d are $\lambda_{d1} = \lambda_{d2} = \cdots = \lambda_{dm} = \epsilon$ and $\lambda_{d,m+1} = \cdots = \lambda_{d,v-1} = 1$. The corresponding orthonormal eigenvectors are denoted as before by $\boldsymbol{\xi}_{di}, 1 \le i \le v-1$, while the normalized eigenvector corresponding to the zero eigenvalue of A_d is $\boldsymbol{\rho}_d/n^{\frac{1}{2}}$. Recalling the definition of the matrix M_{0d} from (2.3.17), we have for a C-design,

$$\begin{aligned} M_{0d} &= R_d^{-1} N_d K_d^{-1} N'_d - \mathbf{1}\mathbf{r}'_d/n \\ &= R_d^{-1}(R_d - C_d) - \mathbf{1}\mathbf{r}'_d/n \\ &= (1-\epsilon)\left(I_v - \mathbf{1}\mathbf{r}'_d/n - R_d^{-\frac{1}{2}} \sum_{i=m+1}^{v-1} \boldsymbol{\xi}_{di}\boldsymbol{\xi}'_{di} R_d^{\frac{1}{2}}\right) \\ &= (1-\epsilon) R_d^{-\frac{1}{2}} \left(I_v - \boldsymbol{\rho}_d\boldsymbol{\rho}'_d/n - \sum_{i=m+1}^{v-1} \boldsymbol{\xi}_{di}\boldsymbol{\xi}'_{di}\right) R_d^{\frac{1}{2}} \\ &= (1-\epsilon) R_d^{-\frac{1}{2}} \left(\sum_{i=1}^{m} \boldsymbol{\xi}_{di}\boldsymbol{\xi}'_{di}\right) R_d^{\frac{1}{2}}. \end{aligned} \tag{4.10.2}$$

Conversely, suppose for a connected design d, the matrix M_{0d} is given by (4.10.2). Also, from (2.3.17), for an arbitrary design d,

$$\begin{aligned} M_{0d} &= R_d^{-1} N_d K_d^{-1} N'_d - \mathbf{1}_v \mathbf{r}'_d/n \\ &= R_d^{-\frac{1}{2}}(I_v - R_d^{-\frac{1}{2}} C_d R_d^{-\frac{1}{2}} - \boldsymbol{\rho}_d\boldsymbol{\rho}'_d/n) R_d^{\frac{1}{2}} \\ &= R_d^{-\frac{1}{2}}(I_v - A_d - \boldsymbol{\rho}_d\boldsymbol{\rho}'_d/n) R_d^{\frac{1}{2}}. \end{aligned} \tag{4.10.3}$$

Equating (4.10.2) and (4.10.3), we have after simplification,

$$A_d = \epsilon \sum_{i=1}^{m} \boldsymbol{\xi}_{di}\boldsymbol{\xi}'_{di} + \sum_{i=m+1}^{v-1} \boldsymbol{\xi}_{di}\boldsymbol{\xi}'_{di}, \tag{4.10.4}$$

which shows that A_d has only two distinct eigenvalues, ϵ and unity.

From (4.10.2), it is easily seen that for every positive integer s,

$$M_{0d}^s = (1-\epsilon)^{s-1} M_{0d}. \tag{4.10.5}$$

The condition in (4.10.5) was used by Saha (1976) to define a C-design. The above analysis shows that equivalently, a connected block design d is a C-design if and only if d has only two distinct canonical efficiency factors, $\epsilon \in (0,1)$ and unity, with respective multiplicities m and $v - 1 - m$. This in turn means that under a C-design, $v-1-m$ basic contrasts (see Section 2.3) are each estimated with full efficiency while the rest are estimated with the same efficiency factor $\epsilon \in (0,1)$. Clearly, an orthogonal design is a C-design with $m = 0$ and a nonorthogonal efficiency-balanced design is a C-design with $m = v - 1$.

It can also be seen that for a C-design d, the matrix M_{0d} can be written as

$$M_{0d} = (1-\epsilon)L, \tag{4.10.6}$$

where

$$L = R_d^{-\frac{1}{2}} \sum_{i=1}^{m} \xi_{di}\xi_{di}' R_d^{\frac{1}{2}} \tag{4.10.7}$$

is an idempotent matrix of rank m. Also, it can be shown that for a C-design d, a choice of a generalized inverse of C_d is

$$\begin{aligned} C_d^- &= R_d^{-\frac{1}{2}} \left(\epsilon^{-1} \sum_{i=1}^{m} \xi_{di}\xi_{di}' + \sum_{i=m+1}^{v-1} \xi_{di}\xi_{di}' \right) R_d^{-\frac{1}{2}} \\ &= (\epsilon^{-1} M_{0d} + I_v) R_d^{-1} - \mathbf{1}_v \mathbf{1}_v'/n. \end{aligned} \tag{4.10.8}$$

A similar g-inverse of C_d was also obtained by Caliński (1971).

As observed earlier, all efficiency-balanced designs are C-designs. However, the class of C-designs is much larger and includes designs that need not be efficiency-balanced. The following result due to Saha (1976) provides a characterization of C-designs, the proof of which is left as an exercise.

Theorem 4.10.1 *An incomplete block design d with incidence matrix N_d is a C-design if and only if there exists a constant $\mu \in [0,1)$ such that*

$$N_d K_d^{-1} V_d K_d^{-1} N_d' = \mathbf{0} \tag{4.10.9}$$

where

$$V_d = N_d' R_d^{-1} N_d - \mu K_d - n^{-1}(1-\mu)\boldsymbol{k}_d \boldsymbol{k}_d'. \tag{4.10.10}$$

The following are consequences of Theorem 4.10.1.

Corollary 4.10.1 *An incomplete block design d is a C-design if for some constant $\mu \in [0, 1)$,*

$$N_d' R_d^{-1} N_d = \mu K_d + n^{-1}(1 - \mu) \boldsymbol{k}_d \boldsymbol{k}_d'. \tag{4.10.11}$$

Corollary 4.10.2 *An incomplete block design d is a C-design if for some constant $\mu \in [0, 1)$,*

$$N_d K_d^{-1} N_d' R_d^{-1} N_d = \mu N_d + n^{-1}(1 - \mu) \boldsymbol{r}_d \boldsymbol{k}_d'. \tag{4.10.12}$$

For some more results on C-designs, see Saha (1976). For methods of construction of C-designs, see Caliński and Kageyama (2000), where more references can be found. The notion of C-designs (or, simple partially efficiency-balanced designs) has been generalized to partially efficiency-balanced (PEB) designs by Puri and Nigam (1977). However, as observed by Pal (1980) and Dey and Gupta (1986), every conceivable connected incomplete block design is a PEB design with m efficiency classes for some integer m, $1 \leq m \leq v - 1$. Thus, it appears that the concept of a PEB design with arbitrary number of efficiency classes is rather weak.

4.11 Alpha Designs

In agricultural and forestry experiments, often efficient resolvable designs with small number of replications are desired. This is because resolvable designs can be conducted with one replication at a time, with different replications applied at different points of time or, even at different locations. If locations or different time periods are used as replications of a resolvable incomplete block design, then the variability among locations for instance, can be eliminated, apart from the variability within a location. Though several designs that have been covered so far have a resolvable solution, there are still some combinations of v, the number of treatments and r, the replication number, for which either a resolvable solution is unknown or the known design has poor efficiency. In view of this, a very flexible class of resolvable incomplete block designs called α designs was introduced by Patterson and Williams (1976a). An α design can be constructed as follows:

The first step towards the construction of an α design is to consider a $k \times r$ array, $\boldsymbol{\alpha}$ with elements from the set of residue classes mod s, where k is the block size, r is the replication of each treatment and $s \geq 2$ is an integer which equals the number of blocks in each replicate. The array α is called the "generating array". Next, each column of α is used to generate $s-1$ more columns by a cyclical development. Let the resulting $k \times rs$ array be denoted by α^*. Finally, add is to each element of the $(i+1)$st row of α^*. The columns of the resulting array are the blocks of the required α design and each set of columns (blocks) generated from the same column of α constitutes a replication.

Example 4.11.1 Suppose it is desired to construct an α design with $v = 12, k = 3, r = 3$ and $s = 4$. One can choose the following array as the generating array:

$$\alpha = \begin{matrix} 0 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 1 \end{matrix}$$

One can now develop each column of the above array mod 4 to get the following intermediate array:

$$\alpha^* = \begin{matrix} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 2 & 3 & 0 & 1 & 3 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 & 3 & 0 & 1 & 2 & 1 & 2 & 3 & 0 \end{matrix}$$

Following the construction method described above, we get the desired α design with the following blocks:

	Block 1	Block 2	Block 3	Block 4
	0	1	2	3
Replicate I	4	5	6	7
	8	9	10	11

	Block 5	Block 6	Block 7	Block 8
	0	1	2	3
Replicate II	6	7	4	5
	11	8	9	10

	Block 9	Block 10	Block 11	Block 12
	0	1	2	3
Replicate III	7	4	5	6
	9	10	11	8

The generating array used in the above example has all elements equal to zero in the first row and first column. Such an array is called a "reduced array". By a suitable relabeling of elements, all arrays can be represented by reduced arrays. This fact is useful in finding efficient α designs as, the search of such designs can be restricted to the generating arrays that are in the reduced form.

Upper bounds for the average efficiency factor of an α design have been given by Patterson and Williams (1976b) and Williams and Patterson (1977). A software, called ALPHA$^+$ providing α designs in the parameter ranges $2 \leq r \leq 10, 2 \leq k \leq 20, v \leq 500$ has been prepared by Williams and Talbot (1993). A more general software package, called CycDesigN has been provided by Whitaker, Williams and John (1997). Both packages provide α designs with high efficiency factors. A monograph on α designs with tables of designs and their efficiencies has recently been prepared by Parsad *et al.* (2007). A short table of generating arrays for constructing α designs can be found in Patterson, Williams and Hunter (1978) and also reproduced in John (1987, p. 86).

4.12 Exercises

4.1. Using (4.3.8), provide an alternative proof of the third relation in (4.2.3).

4.2. Show that a group-divisible design with two-associate classes satisfying $\lambda_2 = 0$ is necessarily disconnected.

4.3. Consider a two-associate PBIB design with v treatments, $r = 2$ replicates and concurrence parameters λ_1, λ_2. Show that if such a design has $\lambda_1 = 2$ and $\lambda_2 = 0$, then the design is disconnected.

4.4. Show that the inequality $rk - v\lambda_2 \geq 0$ holds for singular group-divisible designs as well.

4.5. Show that for a semi-regular group-divisible design, $\lambda_2 - \lambda_1 > 0$.

4.6. Provide a proof of Theorems 4.4.2 and 4.4.4.

4.7. Suppose there exists a two-class association scheme with v treatments satisfying the condition $p_{11}^1 = p_{11}^2$. Show that this implies the existence of a BIB design. Determine the parameters of the BIB design.

4.8. Is it true that the complement of a connected group-divisible design is also connected? If so, provide a proof; otherwise, give a counterexample.

4.9. Construct a group-divisible design with parameters $v = 8 = b, r = 3 = k, \lambda_1 = 0, \lambda_2 = 1, m = 4, n = 2$.

4.10. Give a proof of Theorem 4.4.6.

4.11. Show that a necessary condition for the existence of a partial geometry (r, k, t) is that $\alpha = \dfrac{rk(r-1)(k-1)}{t(k+r-t-1)}$ is a positive integer.

4.12. Provide a proof of Theorem 4.4.17.

4.13. Let $v = 3n$ for some integer n and consider the following $3n$ blocks involving the v treatments, which are labeled as $a_i, b_i, c_i,\ 1 \le i \le n.$:

$$(a_1, a_2, \ldots, a_n, b_i); (b_1, b_2, \ldots, b_n, c_i); (c_1, c_2, \ldots, c_n, a_i), 1 \le i \le n.$$

Show that these $3n$ blocks form a two-associate PBIB design. Determine the underlying association scheme and also the design parameters.

4.14. Consider a linked block design with v treatments, b blocks, replication r and block size k. Show that for such a design, $k(r-1)/(b-1)$ is a positive integer.

4.15. Show that for a two-associate triangular design with parameters $v = m(m-1)/2, b, r, k, \lambda_1 = 0, \lambda_2 > 0$, the inequality $2k \le m$ holds.

4.16. Show that a two-associate triangular design with usual parameters $v = m(m-1)/2, b, r, k, \lambda_1, \lambda_2$ is a linked block design if and only if (i) $r - 2\lambda_1 + \lambda_2 = 0$ and $b = m$ or, (ii) $r - (m-3)\lambda_2 + (m-4)\lambda_1 = 0$ and $b = \dfrac{(m-1)(m-2)}{2}$.

4.17. Let $\mathcal{A}$ be a two-class association scheme with parameters v, n_i, p^i_{js}, $i, j, s = 1, 2$. For the uth treatment $(1 \le u \le v)$, form a block by putting all those treatments that are ith associates of u, $i = 1, 2$. Show that these v blocks form a PBIB design based on $\mathcal{A}$. Determine the parameters of the designs so obtained.

4.18. Let d be a two-associate PBIB design obtained by treating the points of a partial geometry (r, k, t) as treatments and the lines of the geometry as blocks and suppose N_d is the incidence matrix of d. Find the eigenvalues of $N_d N_d'$.

4.19. Let d be a BIB design with parameters $v^*, b^*(> v^*), r^*, k^*, \lambda^* = 1$. Let the blocks of d be treated as treatments, so that we now have $v = b^*$ treatments. On these v treatments, define an association rule as follows: two block labels are first associates if these blocks have one treatment of d in common and, second associates otherwise. Show that this rule is a two-class association scheme and determine its parameters.

4.20. Consider $v = 6$ treatments, labeled $0, 1, \ldots, 5$ on which the following association rule is defined: for treatment i, the first associates are

$(i+1, i+5)$, the second associates are $(i+2, i+4)$ and $i+3$ is the third associate, $0 \leq i \leq 5$, and the sums are reduced modulo 6. Examine whether the above association rule is an association scheme with three classes.

4.21. For $i = 1, 2$, let N_i be the incidence matrix of a BIB design d_i with parameters $v^{(i)}, b^{(i)}, r^{(i)}, k^{(i)}, \lambda^{(i)}$ and let $\bar{N}_i$ be the incidence matrix of the complement of d_i. Show that

$$N = N_1 \otimes N_2 + \bar{N}_1 \otimes \bar{N}_2$$

is the incidence matrix of a rectangular design.

4.22. Provide a proof of Theorem 4.10.1.

Chapter 5

More Incomplete Block Designs

5.1 Introduction

In Chapters 3 and 4 we have considered a variety of incomplete block designs. It was tacitly assumed therein that all treatments are equally important and thus, the interest was primarily in elementary treatment contrasts. However, there are several situations in practice wherein interest shifts from elementary treatment contrasts to contrasts of special types. For instance, in the context of factorial experiments, interest centers around contrasts belonging to the factorial effects, namely those belonging to main effects and interactions. These contrasts of course are quite different from elementary treatment contrasts. Similarly, in the context of biological assays, one is interested in specific contrasts among the dose (treatment) effects that are crucial for estimating the relative potency and making validity tests. In this chapter, we consider incomplete block designs for some specific applications. We consider four experimental situations and discuss incomplete block designs for each of these. Factorial experiments are an extremely important class of experiments with applications in almost all areas, including agriculture, biology, physical and chemical sciences etc. A comprehensive account of designs for factorial experiments is available in Gupta and Mukerjee (1989) and in Section 5.2, we briefly state some important results on incomplete block designs for factorial experiments. In Section 5.3, incomplete block designs for a type of biological assays, called parallel line assays are studied. Incomplete block designs for making comparisons between a set of test treatment and a special treatment, called control are considered in Section 5.4. In Section 5.5, we consider a type of plant breeding experiments, called diallel cross experiments and study incomplete block designs for such experiments. Finally, in Section 5.6, we consider the robustness of incomplete block designs against the non-availability of data or, presence of an outlying observation and trend-free block designs.

5.2 Designs for Factorial Experiments

In this section, we present a brief account of some major concepts and results related to incomplete block designs for factorial experiments. Typically, in such an experiment there is an output variable which is dependent on several controllable or input variables. These input variables are called *factors*. For each factor there are two or more possible settings known as *levels*. Any combination of the levels of all the factors under consideration is called a *treatment combination*. Factorial experiments aim at exploring the effects of the individual factors and their inter–relationship as well and, such effects are called factorial effects.

Factorial experiments and the designs associated with such experiments were introduced by Fisher (1937). Important early contributions were made by Yates (1937), Bose and Kishen (1940), Bose (1947) and Nair and Rao (1941, 1942a, 1948). The mathematical theory underlying incomplete block designs for symmetric factorial experiments of the type p^n, where p is a prime or a prime power was fully developed by Bose (1947) and a brief review of this theory and related results may be found in Dey and Mukerjee (2003). A host of other authors contributed to this and related areas subsequently. For lucid descriptions of the design and analysis of factorial experiments, their applications and recent developments, we refer to Box, Hunter and Hunter (1978), Hinkelmann and Kempthorne (1994, 2005), Raktoe, Hedayat and Federer (1981), Wu and Hamada (2000) and Mukerjee and Wu (2006).

In the context of factorial experiments, interest lies in treatment contrasts belonging to factorial effects. It is therefore important to define such contrasts. Consider a factorial experiment involving $n(\geq 2)$ factors, say $F_1, \ldots, F_n$. For $1 \leq i \leq n$, let the factor F_i have $m_i(\geq 2)$ distinct levels. An experiment of this kind is called an $m_1 \times \cdots \times m_n$ factorial experiment. If $m_1 = \cdots = m_n = m$, say, then the set up is that of a *symmetric* factorial experiment; otherwise, we have a *mixed* or, *asymmetric* factorial experiment. For $1 \leq i \leq n$, let the levels of the factor F_i be coded as $0, 1, \ldots, m_i - 1$. A typical treatment combination can be expressed as an ordered n-tuple $j_1 j_2 \ldots j_n$ and the fixed effect of $j_1 j_2 \ldots j_n$ is denoted by $\tau(j_1 \ldots j_n)$, $0 \leq j_i \leq m_i - 1, 1 \leq i \leq n$.

Clearly, the total number of treatment combinations in an $m_1 \times \cdots \times m_n$ experiment is $v = \prod_{i=1}^{n} m_i$. Henceforth, it is assumed that these v treatment combinations are lexicographically ordered. We let $\mathcal{V}$ to denote the set of v treatment combinations and $\boldsymbol{\tau}$ to denote the $v \times 1$ vector of treatment effects $\tau(j_1 \dots j_n)$, arranged lexicographically.

The treatment effects, i.e., the elements of $\boldsymbol{\tau}$ are unknown parameters. A linear parametric function

$$\sum \cdots \sum l(j_1 \dots j_n)\tau(j_1 \dots j_n), \tag{5.2.1}$$

where $\{l(j_1 \dots j_n)\}$ are real numbers, not all zero simultaneously, such that $\sum \cdots \sum l(j_1 \dots j_n) = 0$ and the summation extends over $j_1 \dots j_n \in \mathcal{V}$, is called a treatment contrast. In factorial experiments, one is interested in special types of treatment contrasts, namely those belonging to factorial effects. Following Bose (1947), a treatment contrast of the type (5.2.1) is said to belong to the factorial effect $F_{i_1}F_{i_2}\dots F_{i_g}$ $(1 \le i_1 < i_2 < \cdots < i_g \le n, 1 \le g \le n)$ if the following conditions hold:

(i) $l(j_1 \dots j_n)$ depends only on $j_{i_1} \dots j_{i_g}$, and

(ii) writing $l(j_1 \dots j_n) = \bar{l}(j_{i_1} \dots j_{i_g})$ by virtue of (i) above, the sum of $\bar{l}(j_{i_1} \dots j_{i_g})$ separately over each of the arguments $j_{i_1}, \dots, j_{i_g}$ is zero.

By (i) and (ii), there are $\prod_{u=1}^{g}(m_{i_u} - 1)$ linearly independent contrasts belonging to the factorial effect $F_{i_1} \dots F_{i_g}$. A factorial effect is called a *main effect* if it involves just one factor $(g = 1)$ and an *interaction* if $g > 1$. The total number of factorial effects is $2^n - 1$, there being n main effects, $\binom{n}{2}$ 2-factor interactions, $\binom{n}{3}$ 3-factor interactions, etc. In the absence of a better notation, the ith factor and its main effect will both be denoted by F_i.

Kurkjian and Zelen (1962, 1963) introduced a tensor calculus for factorial experiments which provides a powerful tool for the analysis of factorial experiments. This calculus in particular helps in expressing the notation in a compact and convenient form and has been effectively used, among others, by Gupta and Mukerjee (1989), Dey and Mukerjee (1999) and Mukerjee and Wu (2006). In what follows, we briefly review the essentials of this calculus in the context of an $m_1 \times \cdots \times m_n$ factorial experiment.

Let Ω denote the set of all binary n-tuples and $\Omega^* = \Omega \setminus \{00\dots0\}$. For each $\boldsymbol{x} = x_1x_2\dots x_n \in \Omega$, define

$$\alpha(\boldsymbol{x}) = \prod_{i=1}^{n}(m_i - 1)^{x_i}. \tag{5.2.2}$$

It is easy to see that there is a 1-1 correspondence between Ω^* and the set of factorial effects in the sense that a typical factorial effect $F_{i_1} \dots F_{i_g}$ corresponds to the element $\boldsymbol{x} = x_1 x_2 \dots x_n$ of Ω^* such that $x_{i_1} = x_{i_2} = \dots = x_{i_g} = 1$ and all other x_i's are zero. Thus the $2^n - 1$ factorial effects can be represented by $F^{\boldsymbol{x}}$, $\boldsymbol{x} \in \Omega^*$. For instance, with $n = 2$, $\Omega = \{00, 10, 01, 11\}$, $\Omega^* = \{10, 01, 11\}$, the main effect of the first factor can be represented as F^{10}, that of the second factor as F^{01} and the 2-factor interaction as F^{11}. As noted earlier, for each $\boldsymbol{x} \in \Omega^*$, there are $\alpha(\boldsymbol{x})$ linearly independent treatment contrasts belonging to the factorial effect $F^{\boldsymbol{x}}$, where $\alpha(\boldsymbol{x})$ is given by (5.2.2).

For $1 \le i \le n$, let P_i be an $(m_i - 1) \times m_i$ matrix such that the matrix $(m_i^{-\frac{1}{2}} \mathbf{1}_{m_i}, P_i')$ is orthogonal. It follows then that

$$P_i \mathbf{1}_{m_i} = \mathbf{0}, \; P_i P_i' = I_{m_i - 1}. \tag{5.2.3}$$

For example, if $n = 2, m_1 = 2, m_2 = 3$, one can take

$$P_1 = \left(\frac{1}{\sqrt{2}} \; \frac{-1}{\sqrt{2}} \right), \; P_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}. \tag{5.2.4}$$

Note that the matrices P_i satisfying (5.2.3) are nonunique in general; however, the main ideas and conclusions that follow do not depend on the specific choice of the P_i's – see Remark 2.3.1 in Dey and Mukerjee (1999) for details. For each $\boldsymbol{x} = x_1 \dots x_n \in \Omega$, define the $\alpha(\boldsymbol{x}) \times v$ matrix

$$P^{\boldsymbol{x}} = P_1^{x_1} \otimes \dots \otimes P_n^{x_n} = \bigotimes_{i=1}^{n} P_i^{x_i}, \tag{5.2.5}$$

where for $1 \le i \le n$,

$$P_i^{x_i} = \begin{cases} m_i^{-\frac{1}{2}} \mathbf{1}'_{m_i} & \text{if } x_i = 0, \\ P_i & \text{if } x_i = 1. \end{cases} \tag{5.2.6}$$

One then has the following result.

Lemma 5.2.1 *For each $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, $\boldsymbol{x} \neq \boldsymbol{y}$,*
(a) $P^{\boldsymbol{x}} (P^{\boldsymbol{x}})' = I_{\alpha(\boldsymbol{x})}$,
(b) $P^{\boldsymbol{x}} (P^{\boldsymbol{y}})' = \mathbf{0}$.

It is easy to observe that for each $\boldsymbol{x} \in \Omega^*$, the elements of $P^{\boldsymbol{x}} \boldsymbol{\tau}$ are treatment contrasts belonging to the factorial effect $F^{\boldsymbol{x}}$. Also, by part (a) of Lemma 5.2.1, $Rank(P^{\boldsymbol{x}}) (= \alpha(\boldsymbol{x}))$ equals the number of linearly independent treatment contrasts belonging to $F^{\boldsymbol{x}}$. The following result, giving a convenient representation of contrasts belonging to factorial effects, is now immediate.

Lemma 5.2.2 *For each $\boldsymbol{x} \in \Omega^*$, the elements of $P^{\boldsymbol{x}}\boldsymbol{\tau}$ represent a complete set of orthonormal contrasts belonging to the factorial effect $F^{\boldsymbol{x}}$. Furthermore, contrasts belonging to different factorial effects are mutually orthogonal.*

It is also possible to give an interpretation of $P^{00\ldots0}\boldsymbol{\tau}$. By (5.2.5) and (5.2.6), $P^{00\ldots0} = v^{-\frac{1}{2}}\mathbf{1}_v'$ and thus, $P^{00\ldots0}\boldsymbol{\tau} = v^{\frac{1}{2}}\bar{\tau}$, where $\bar{\tau}$ is the arithmetic mean of the quantities $\{\tau(j_1 \ldots j_n)\}$.

Remark 5.2.1 There is an equivalent way of representation of treatment contrasts belonging to the factorial effects in the context of an $m_1 \times \cdots \times m_n$ factorial experiment. For each $\boldsymbol{x} = (x_1, \ldots, x_n) \in \Omega^*$, let

$$M^{\boldsymbol{x}} = M_1^{x_1} \otimes M_2^{x_2} \otimes \cdots \otimes M_n^{x_n} = \bigotimes_{i=1}^{n} M_i^{x_i}, \tag{5.2.7}$$

where for $1 \leq i \leq n$,

$$M_i^{x_i} = \begin{cases} I_{m_i} - m_i^{-1}J_{m_i} & \text{if } x_i = 1, \\ m_i^{-1}J_{m_i} & \text{if } x_i = 0. \end{cases} \tag{5.2.8}$$

Then, analogous to Lemma 5.2.2, it can be shown that (i) the elements of $M^{\boldsymbol{x}}\boldsymbol{\tau}$ represent a complete set of treatment contrasts belonging to $F^{\boldsymbol{x}}$ and, (ii) treatment contrasts belonging to any two distinct factorial effects are mutually orthogonal. Since for each i, $1 \leq i \leq n$, $(P_i^{x_i})'P_i^{x_i} = M_i^{x_i}$ irrespective of whether $x_i = 0$ or 1, we have for every $\boldsymbol{x} \in \Omega^*$,

$$(P^{\boldsymbol{x}})'P^{\boldsymbol{x}} = M^{\boldsymbol{x}} \tag{5.2.9}$$

and in this sense, the two representations are equivalent.

Consider again the set up of an $m_1 \times \cdots \times m_n$ factorial and suppose the $v = \prod_{i=1}^{n} m_i$ treatment combinations are laid out in an incomplete block design d involving b blocks of sizes $k_{d1}, \ldots, k_{db}$. Also, for $1 \leq i \leq v$, let r_{di} denote the replication of the ith treatment combination. Recall from Chapter 2 that the reduced normal equations for estimating linear functions of the elements of $\boldsymbol{\tau}$ is $C_d\boldsymbol{\tau} = \boldsymbol{Q}$, where $\boldsymbol{\tau}$ is the vector of the effects of the treatment combinations, as defined in the beginning of this section.

As emphasized earlier, in the context of factorial experiments, one is interested in drawing inferences on contrasts belonging to factorial effects and in such a scenario, great simplification occurs in the interpretation of results if the incomplete block design under consideration has *orthogonal factorial structure.* We begin with a formal definition where the term "design" refers to an incomplete block design of the type considered above.

Definition 5.2.1 *A design d will be said to have the orthogonal factorial structure (OFS) if under d, the BLUEs of estimable treatment contrasts belonging to distinct factorial effects are mutually uncorrelated.*

The above definition implies that OFS holds for a design if for each pair $\boldsymbol{x}, \boldsymbol{y} \in \Omega^*$ $\boldsymbol{x} \neq \boldsymbol{y}$, the BLUE of every estimable linear combination of the elements of $P^{\boldsymbol{x}}\boldsymbol{\tau}$ is uncorrelated with the BLUE of every estimable linear combination of the elements of $P^{\boldsymbol{y}}\boldsymbol{\tau}$. When a connected design has OFS, the adjusted treatment sum of squares can be split up orthogonally into components due to different factorial effects and, as such, these components can be shown in the same analysis of variance table. Clearly, OFS is an important and desirable property of a factorial design.

Another important concept in the set up of designs for a factorial experiment is that of *balance.* We define this notion following Shah (1958).

Definition 5.2.2 *In a design d, a factorial effect $F^{\boldsymbol{x}}$, $\boldsymbol{x} \in \Omega^*$ is said to be balanced if either*
(a) all treatment contrasts belonging to $F^{\boldsymbol{x}}$ are estimable under d and the BLUEs of all normalized contrasts belonging to $F^{\boldsymbol{x}}$ have the same variance, or,
(b) no contrast belonging to $F^{\boldsymbol{x}}$ is estimable under d.

A design d is called balanced if $F^{\boldsymbol{x}}$ is balanced for each $\boldsymbol{x} \in \Omega^*$. In Definition 5.2.2, the trivial situation (b) is included merely for mathematical completeness. Note that the situation in (b) will never arise if the design under consideration is connected. The following result provides a characterization of balanced designs.

Lemma 5.2.3 *In a design d, a factorial effect $F^{\boldsymbol{x}}$ is balanced in the sense (a) of Definition 5.2.2 if and only if all treatment contrasts belonging to $F^{\boldsymbol{x}}$ are estimable under d and the BLUEs of every pair of mutually orthogonal contrasts belonging to $F^{\boldsymbol{x}}$ are uncorrelated.*

Proof. First, assume that d is balanced in the sense (a) of Definition 5.2.2. Let $\boldsymbol{p}'_1\boldsymbol{\tau}$ and $\boldsymbol{p}'_2\boldsymbol{\tau}$ be two mutually orthogonal contrasts belonging to $F^{\boldsymbol{x}}$. Set

$$\begin{aligned} \boldsymbol{\xi}_i &= \boldsymbol{p}_i/\sqrt{\boldsymbol{p}_i'\boldsymbol{p}_i},\ i=1,2, \\ \boldsymbol{\xi} &= (\boldsymbol{\xi}_1+\boldsymbol{\xi}_2)/\sqrt{2}. \end{aligned} \tag{5.2.10}$$

Then it is easy to see that $\boldsymbol{\xi}_1'\boldsymbol{\tau}, \boldsymbol{\xi}_2'\boldsymbol{\tau}$ and $\boldsymbol{\xi}'\boldsymbol{\tau}$ are each a normalized treatment contrast belonging to $F^{\boldsymbol{x}}$. By the hypothesis, $F^{\boldsymbol{x}}$ is balanced and thus

$$\operatorname{Var}(\boldsymbol{\xi}'\hat{\boldsymbol{\tau}}) = \operatorname{Var}(\boldsymbol{\xi}_1'\hat{\boldsymbol{\tau}}) = \operatorname{Var}(\boldsymbol{\xi}_2'\hat{\boldsymbol{\tau}}), \tag{5.2.11}$$

where $\hat{\boldsymbol{\tau}}$ is a solution of the reduced intra-block normal equations. Also,

$$\begin{aligned} \operatorname{Var}(\boldsymbol{\xi}'\hat{\boldsymbol{\tau}}) &= \frac{1}{2}\operatorname{Var}(\boldsymbol{\xi}_1'\hat{\boldsymbol{\tau}}+\boldsymbol{\xi}_2'\hat{\boldsymbol{\tau}}) \\ &= \frac{1}{2}[\operatorname{Var}(\boldsymbol{\xi}_1'\hat{\boldsymbol{\tau}})+\operatorname{Var}(\boldsymbol{\xi}_2'\hat{\boldsymbol{\tau}})+2\operatorname{Cov}(\boldsymbol{\xi}_1'\hat{\boldsymbol{\tau}},\boldsymbol{\xi}_2'\hat{\boldsymbol{\tau}})]. \end{aligned}$$

This, by virtue of (5.2.11), gives $\operatorname{Cov}(\boldsymbol{\xi}_1'\hat{\boldsymbol{\tau}},\boldsymbol{\xi}_2'\hat{\boldsymbol{\tau}}) = 0$ and, hence $\operatorname{Cov}(\boldsymbol{p}'_1\hat{\boldsymbol{\tau}},\boldsymbol{p}'_2\hat{\boldsymbol{\tau}}) = 0$. This proves the necessity.

To prove the converse, consider two distinct normalized contrasts $\boldsymbol{p}'_1\boldsymbol{\tau}$ and $\boldsymbol{p}'_2\boldsymbol{\tau}$, each belonging to the factorial effect $F^{\boldsymbol{x}}$. If $\boldsymbol{p}_1 = -\boldsymbol{p}_2$, then trivially, $\operatorname{Var}(\boldsymbol{p}'_1\hat{\boldsymbol{\tau}}) = \operatorname{Var}(\boldsymbol{p}'_2\hat{\boldsymbol{\tau}})$. So, assume that $\boldsymbol{p}_1 \neq -\boldsymbol{p}_2$. Then, $(\boldsymbol{p}_1+\boldsymbol{p}_2)'\boldsymbol{\tau}$ and $(\boldsymbol{p}_1-\boldsymbol{p}_2)'\boldsymbol{\tau}$ are mutually orthogonal contrasts belonging to $F^{\boldsymbol{x}}$ and hence, under the stated condition in the lemma,

$$\operatorname{Cov}((\boldsymbol{p}_1+\boldsymbol{p}_2)'\hat{\boldsymbol{\tau}},(\boldsymbol{p}_1-\boldsymbol{p}_2)'\hat{\boldsymbol{\tau}}) = 0,$$

which yields $\operatorname{Var}(\boldsymbol{p}'_1\hat{\boldsymbol{\tau}}) = \operatorname{Var}(\boldsymbol{p}'_2\hat{\boldsymbol{\tau}})$. □

The next result is a consequence of Lemmas 5.2.2 and 5.2.3.

Corollary 5.2.1 *In a design, a factorial effect $F^{\boldsymbol{x}}$ is balanced in the sense (a) of Definition* 5.2.2 *if and only if all treatment contrasts belonging to $F^{\boldsymbol{x}}$ are estimable and the dispersion matrix of $P^{\boldsymbol{x}}\hat{\boldsymbol{\tau}}$ is a scalar multiple of an identity matrix.*

From the preceding discussion, it follows that the property of OFS ensures "between effect" orthogonality (sometimes called *inter-effect orthogonality*) while, by Lemma 5.2.3, the property of balance ensures "within effect" (*intra-effect*) orthogonality. Therefore, if a design has both OFS and balance, then further simplifications in the analysis and interpretation are achieved. Designs having both OFS and balance are called balanced factorial designs (*cf.* Shah (1958, 1960a)). In view of the above, it becomes important to characterize balanced factorial designs. To achieve that goal, we first introduce some further notation. We continue to consider the set up of an $m_1 \times \cdots \times m_n$ factorial.

For each $\boldsymbol{x} = x_1 \ldots x_n \in \Omega$, let

$$Z^{\boldsymbol{x}} = \bigotimes_{i=1}^{n} Z_i^{x_i}, \tag{5.2.12}$$

where for $1 \leq i \leq n$,

$$Z_i^{x_i} = \begin{cases} I_{m_i} & \text{if } x_1 = 1, \\ J_{m_i} & \text{if } x_i = 0. \end{cases} \tag{5.2.13}$$

We also need the following definition.

Definition 5.2.3 *A $v \times v$ matrix G is said to have property (A) if it can written as*

$$G = \sum_{\boldsymbol{x} \in \Omega} h(\boldsymbol{x}) Z^{\boldsymbol{x}},$$

where $\{h(\boldsymbol{x})\}$, $\boldsymbol{x} \in \Omega$ are real numbers and, as before, $v = \prod m_i$.

We now have the following result, whose proof can be found in Gupta and Mukerjee (1989).

Lemma 5.2.4 *A connected design d is a balanced factorial design (i.e., has both OFS and balance) if and only if the C-matrix of the design can be expressed as*

$$C_d = \sum_{\boldsymbol{x} \in \Omega^*} \nu(\boldsymbol{x}) M^{\boldsymbol{x}} = \sum_{\boldsymbol{x} \in \Omega^*} \nu(\boldsymbol{x}) (P^{\boldsymbol{x}})' P^{\boldsymbol{x}}, \tag{5.2.14}$$

where $\{\nu(\boldsymbol{x})\}, \boldsymbol{x} \in \Omega^*$ are real numbers.

The next result gives a characterization of balanced factorial designs in terms of the property (A) and we again refer to Gupta and Mukerjee (1989) for a proof.

Theorem 5.2.1 *A connected design d has balance and OFS if and only if C_d, the C-matrix of d, has property (A).*

Henceforth, we shall say that a design possesses property (A) if its C-matrix has property (A). As we shall see presently, designs having property (A) have a simple analysis and therefore might be appealing to the experimenters.

Consider a connected design having property (A). Then, for $y \in \Omega^*$, the BLUE of $P^y\tau$ and the dispersion matrix of this estimator can be seen to be given, respectively, by

$$\begin{aligned} P^y\hat{\tau} &= \frac{1}{\nu(y)}P^yQ, \\ \mathbb{D}(P^y\hat{\tau}) &= \sigma^2 I_{\alpha(y)}/\nu(y), \end{aligned}$$

where, as in Chapter 2, Q is the vector of adjusted treatment totals and $\nu(y) > 0$ is a scalar.

The sum of squares (SS) due to the factorial effect F^y, $y \in \Omega^*$ is then given by

$$\begin{aligned} \text{SS due to } F^y &= \text{SS due to } P^y\hat{\tau} \\ &= (P^y\hat{\tau})'(\sigma^{-2}\mathbb{D}(P^y\hat{\tau}))^{-1}(P^y\hat{\tau}) \\ &= \frac{1}{\nu(y)}Q'(P^y)'P^yQ \\ &= \frac{1}{\nu(y)}Q'M^yQ. \end{aligned} \tag{5.2.15}$$

The above formula is extremely simple as it does not involve the inversion of matrices.

Often the designs that are used in practice are either equireplicate or proper or both. For an equireplicate design d with common replication number r say, following the notations of Chapter 2, we have

$$\begin{aligned} C_d &= rI_v - N_dK_d^{-1}N_d' \\ &= r\left(\bigotimes_{i=1}^{n} I_{m_i}\right) - N_dK_d^{-1}N_d' \end{aligned}$$

and thus, C_d has property (A) if and only if $N_dK_d^{-1}N_d'$ has the same property. Similarly, if an equireplicate design d is proper also with common block size k, then

$$C_d = r\left(\bigotimes_{i=1}^{n} I_{m_i}\right) - k^{-1}N_dN_d'$$

so that in such a case C_d has property (A) if and only if the matrix N_dN_d' has property (A). Using these facts and Theorem 5.2.1, we have the following result.

Theorem 5.2.2 *An equireplicate design d has balance and OFS if and only if the matrix $N_dK_d^{-1}N_d'$ has property (A). Furthermore, if d is also proper, then the design has balance and OFS if and only if the matrix N_dN_d' has property (A).*

If the design under consideration is equireplicate and has property (A), then one can provide a simple formula for the efficiency factor of a factorial effect. Let $F^{\boldsymbol{y}}, \boldsymbol{y} \in \Omega^*$ be a factorial effect. Under an equireplicate design d having property (A), we have $\mathbb{D}(P^{\boldsymbol{y}}\hat{\tau}) = \frac{1}{\nu(\boldsymbol{y})}\sigma^2 I_{\alpha(\boldsymbol{y})}$ while under a randomized complete block design with replication r, we have $\mathbb{D}(P^{\boldsymbol{y}}\hat{\tau}) = r^{-1}\sigma^2 I_{\alpha(\boldsymbol{y})}$. Hence the efficiency factor of the factorial effect under the design d is

$$\epsilon(\boldsymbol{y}) = \nu(\boldsymbol{y})/r, \ \boldsymbol{y} \in \Omega^*. \tag{5.2.16}$$

There is a combinatorial characterization of designs which are balanced and have OFS. This characterization is based on the extended group divisible (EGD) association scheme considered in Chapter 4. We state the following two results in this context.

Theorem 5.2.3 *A binary, proper, equireplicate incomplete block design d with incidence matrix N_d is an EGD design if and only if N_dN_d' has property (A).*

Theorem 5.2.4 *A connected, equireplicate, proper and binary design has both balance and OFS if and only if the design is an EGD design.*

For proofs of these results, we refer to Paik and Federer (1973) and Gupta and Mukerjee (1989).

Various methods of construction of balanced designs with OFS have been considered in the literature. For instance, for symmetric factorial experiments with number of levels being a prime or a prime power, such designs can be constructed using the methods described by Bose and Kishen (1940) and Bose (1947) by suitably choosing a confounding scheme. One can refer to Hinkelmann and Kempthorne (2005) for more details on these. Orthogonal arrays of strength two were used by Nair and Rao (1948) to construct EGD designs for an $m_1 \times m_2$ experiment in blocks of size m_1 or m_2. Subsequently, several others provided designs with balance and OFS for experiments of various types; see e.g., Thomson and Dick (1951), Rao (1956), Kramer and Bradley (1957), Kishen (1958), Zelen (1958), Kishen and Srivastava (1959), Das (1960), Shah

(1960b), White and Hultquist (1965), Muller (1966), Puri and Nigam (1976, 1978) and Suen and Chakravarty (1986). We refer to the original sources for details on these and related methods. Several other results on block designs for factorial experiments, including those on construction, can be found e.g., in John and Smith (1972), Cotter, John and Smith (1973), John (1973, 1981), Worthley and Banerjee (1974), Patterson (1976), Bailey (1977), Bailey, Gilchrist and Patterson (1977), Sihota and Banerjee (1981), Mukerjee (1979, 1980, 1981, 1982), Mukerjee and Dean (1986), Mukerjee and Bose (1988a, b), Lewis and Dean (1984, 1985), Lewis, Dean and Lewis (1983), Voss (1986, 1988) and Voss and Dean (1987).

5.3 Designs for Parallel Line Assays

5.3.1 Introducing Bioassays

Biological assays (bioassays) are experiments used for estimating the strength of a substance (stimulus), which could be, for example, a drug, a hormone or a vitamin. This is achieved by comparing two sets of doses, one from material of known strength, called the *standard* preparation and the other from material of unknown strength, called the *test* preparation such that they produce the same response on living organisms like animals, plants or tissues. Thus, if x_s and x_t are the doses of standard and test preparation that produce the same response, then a comparison is made between these two preparations by computing a quantity called *relative potency*, given by $\rho = x_s/x_t$.

Depending on the nature of the preparations, bioassays are conveniently classified into two categories. If the preparations involved in the assay contain the same effective ingredient responsible for producing the response and all other substances that might be present in the preparations are totally inert, then such an assay is called an analytical dilution assay. In contrast to the analytical dilution assays, one can have comparative dilution assays wherein the preparations involved contain different substances producing similar kind of response. Throughout we consider only the analytical dilution assays. In some situations, the stimulus is such that the response is observable almost immediately after the administration of the dose and the response is measurable directly. Such assays are called direct assays. However, in most situations, the dose responsible for a specific response is not directly measurable as soon as as the response occurs and one has to take recourse to indirect methods.

This is achieved via a dose-response relationship. In this section, we will be primarily concerned with indirect analytical dilution assays based on quantitative response. We might add that not all bioassays result in quantitative responses and in fact, in many situations, the response is quantal wherein the experimenter is able to record whether or not each subject manifests a certain recognizable response like death. We however do not consider assays based on quantal responses. For an excellent description of statistical issues in bioassays, see Finney (1978).

5.3.2 Contrasts for Parallel Line Assays

As stated earlier in this section, throughout we consider indirect analytical dilution assays based on quantitative response. The main interest lies in estimating the relative potency defined as the ratio of equivalent doses of the two preparations. Clearly, the relative potency is meaningful when the ratio remains the same over all possible pairs of equivalent doses. Thus it is important to test for the constancy of the ratio of equivalent doses before one undertakes the estimation of the relative potency. These tests are commonly known as validity tests. In what follows, we shall be concerned with *parallel line assays* only and to begin with, we discuss some preliminary issues concerning such assays, following Gupta and Mukerjee (1996).

Consider an indirect assay based on quantitative responses and let s and t denote typical doses of the standard and test preparations respectively and with $x = \log s, z = \log t$, let $\xi_1(x)$ and $\xi_2(z)$ be their respective effects, where log means natural logarithm. Suppose there are m_1 doses of the standard preparation, say $s_1, s_2, \ldots, s_{m_1}$ and m_2 doses of the test preparation, say $t_1, t_2, \ldots, t_{m_2}$. Each $m_i, i = 1, 2$ is at least two. When $m_1 = m_2$, the assay is referred as a *symmetric assay*; otherwise, the assay is called *asymmetric*. It is assumed that the doses are equispaced on the log scale, the common ratio being the same for both the preparations, i.e.,

$$s_i = d_1 u^{i-1},\ 1 \leq i \leq m_1,\ t_i = d_2 u^{i-1},\ 1 \leq i \leq m_2, \qquad (5.3.1)$$

where d_1, d_2 and $u > 1$ are positive constants. It follows then that $0 < s_1 < s_2 < \cdots < s_{m_1}$ and $0 < t_1 < t_2 < \cdots < t_{m_2}$. Note that the integers m_1, m_2 as also the doses $\{s_i\}$ and $\{t_i\}$ are prespecified. There are in all $v = m_1 + m_2$ doses and these doses act as treatments of the experiment. For $1 \leq i \leq m_1$, let τ_i denote the effect of the dose s_i of the standard preparation and similarly, for $1 \le i \le m_2$, let τ_{m_1+i} denote the treatment effect of the dose t_i of the test preparation. Then we have

$$\tau_i = \xi_1(x_i),\ 1 \le i \le m_1,\ \tau_{m_1+i} = \xi_2(z_i),\ 1 \le i \le m_2, \tag{5.3.2}$$

where

$$\begin{aligned} x_i &= \log s_i = \log d_1 + (i-1)\log u,\ 1 \le i \le m_1, \\ z_i &= \log t_i = \log d_2 + (i-1)\log u,\ 1 \le i \le m_2. \end{aligned} \tag{5.3.3}$$

Let

$$\begin{aligned} \boldsymbol{\tau}_1 &= (\tau_1, \ldots, \tau_{m_1})', \\ \boldsymbol{\tau}_2 &= (\tau_{m_1+1}, \ldots, \tau_{m_1+m_2})' \\ \boldsymbol{\tau} &= (\tau_1, \ldots, \tau_v)'. \end{aligned} \tag{5.3.4}$$

The next step is to have an orthogonal polynomial model for $\xi_1(\cdot)$ and $\xi_2(\cdot)$. Let $\psi_{1j}(\cdot), 0 \le j \le m_1 - 1$, represent orthogonal polynomials of degrees $0, 1, \ldots, m_1 - 1$, based on $x_1, \ldots, x_{m_1}$ and similarly, let $\psi_{2j}(\cdot), 0 \le j \le m_2 - 1$, represent orthogonal polynomials of degrees $0, 1, \ldots, m_2 - 1$, based on $z_1, \ldots, z_{m_2}$. That is, $\psi_{1j}(\cdot)$ for instance, is a polynomial of degree j and

$$\sum_{i=1}^{m_1} \psi_{1j}(x_i)\psi_{1k}(x_i) = 0,\ 0 \le j, k \le m_1 - 1, j \ne k. \tag{5.3.5}$$

Since $x_1, \ldots, x_{m_1}$ and $z_1, \ldots, z_{m_2}$ are equispaced, we may take in particular

$$\psi_{i0} \equiv 1,\ \psi_{i1}(h) = h - \log d_i - \frac{m_i - 1}{2}\log u,\ i = 1, 2, \tag{5.3.6}$$

and

$$\psi_{i2}(h) = \{\psi_{i1}(h)\}^2 - \frac{m_i^2 - 1}{12}(\log u)^2,\ i = 1, 2, \tag{5.3.7}$$

where the last expression is valid only for $m_1, m_2 \ge 3$. Consider now the models

$$\xi_1(x) = \sum_{j=0}^{m_1-1} \beta_{1j}\psi_{1j}(x),\quad \xi_2(z) = \sum_{j=0}^{m_2-1} \beta_{2j}\psi_{2j}(z), \tag{5.3.8}$$

where the $\{\beta_{1j}\}$ and $\{\beta_{2j}\}$ are unknown parameters. These parameters will subsequently be expressed in terms of the treatment (dose) effects $\tau_1, \ldots, \tau_v$.

In a parallel line assay, interest lies in estimating the relative potency under the assumption that $\xi_1(x)$ and $\xi_2(z)$ can be represented by *parallel* straight lines, the validity of this assumption being tested before estimating the relative potency. This assumption is equivalent to the following:

$$\beta_{11} = \beta_{21}, \tag{5.3.9}$$

$$\beta_{1j} = 0,\ 2 \le j \le m_1 - 1,\ \beta_{2j} = 0,\ 2 \le j \le m_2 - 1. \tag{5.3.10}$$

The relations in (5.3.10) indicate the linearity of $\xi_1(x)$ and $\xi_2(z)$ whereas (5.3.9) signifies the parallelism of such linear relations. If (5.3.9) and (5.3.10) hold and if we define $\beta_1 = \dfrac{\beta_{11}+\beta_{21}}{2}$ as the common slope of the two straight lines, then we have

$$\begin{aligned}
\xi_1(x) &= \beta_{10} + \beta_1\left(x - \log d_1 - \frac{m_1-1}{2}\log u\right),\\
\xi_2(z) &= \beta_{20} + \beta_1\left(z - \log d_2 - \frac{m_2-1}{2}\log u\right).
\end{aligned}$$

Thus, under (5.3.9) and (5.3.10), two doses s and t of the standard and test preparations, respectively, are equivalent if and only if

$$\begin{aligned}
&\beta_{10} + \beta_1\left(x - \log d_1 - \frac{m_1-1}{2}\log u\right)\\
=\ &\beta_{20} + \beta_1\left(z - \log d_2 - \frac{m_2-1}{2}\log u\right),
\end{aligned}$$

where $x = \log s, z = \log t$. The relative potency, ρ of the test preparation, relative to the standard can then be expressed as

$$\begin{aligned}
\rho = s/t &= \exp(x-z)\\
&= (d_1/d_2)u^{(m_1-m_2)/2}\exp\{-(\beta_{10}-\beta_{20})/\beta_1\},
\end{aligned} \tag{5.3.11}$$

which is a constant for all pairs (s,t) of equivalent doses. From (5.3.9)–(5.3.11), we see that in the context of parallel line assays, the quantities $\beta_{10} - \beta_{20}$, $\beta_1 = (\beta_{11}+\beta_{21})/2, (\beta_{11}-\beta_{21})/2$ and $\beta_{1j}, \beta_{2j},\ j \ge 2$ are of special relevance. The relative potency is a function of the first two while the ignorability of the rest is equivalent to the assumption that the assay is valid, i.e., under which the relative potency is meaningful. Now, from (5.3.2) and (5.3.8), $\tau_i = \xi_1(x_i) = \sum_{j=0}^{m_1-1}\beta_{1j}\psi_{1j}(x_i), 1 \le i \le m_1$, so that for $0 \le j \le m_1 - 1$, we have

$$\beta_{1j} = \frac{\sum_{i=1}^{m_1}\psi_{1j}(x_i)\tau_i}{\sum_{i=1}^{m_1}\{\psi_{1j}(x_i)\}^2}. \tag{5.3.12}$$

Similarly, for $0 \le j \le m_2 - 1$, we have

$$\beta_{2j} = \frac{\sum_{i=1}^{m_2} \psi_{2j}(z_i)\tau_{m_1+i}}{\sum_{i=1}^{m_2}\{\psi_{2j}(z_i)\}^2}. \tag{5.3.13}$$

In particular, β_{11} and β_{21} are given by

$$\begin{aligned}
\beta_{11} &= \frac{12}{\log u}\left\{\sum_{i=1}^{m_1}\left(i - \frac{m_1+1}{2}\right)\tau_i\right\} / \{m_1(m_1^2-1)\}, \\
\beta_{21} &= \frac{12}{\log u}\left\{\sum_{i=1}^{m_2}\left(i - \frac{m_2+1}{2}\right)\tau_{m_1+i}\right\} / \{m_2(m_2^2-1)\}.
\end{aligned}$$

It can be seen that each of β_{11} and β_{21} is a contrast among $\tau_1, \ldots, \tau_v$ and thus, a weighted average among these is also a contrast among $\tau_1, \ldots, \tau_v$. Similarly, it can be seen that $\beta_{10} - \beta_{20}$, $\beta_{11} - \beta_{21}$ and β_{ij}, $i = 1, 2, j \ge 2$ are each a contrast among $\tau_1, \ldots, \tau_v$. Hence, if the experimental design used for a parallel line assay ensures the estimability (and, hence testability) of all treatment contrasts then on the basis of the data one might test the significance of the contrasts representing $(\beta_{11} - \beta_{21})/2, \beta_{1j}, \beta_{2j}, j \ge 2$ and on acceptance of hypotheses that each one is zero, proceed to estimate ρ by replacing $\beta_{10} - \beta_{20}$ and β_1 by their respective best linear unbiased estimators. We may now denote

$$\theta_p = \beta_{10} - \beta_{20},\ \theta_1 = \beta_1,\ \theta_1' = \beta_{11} - \beta_{21}, \tag{5.3.14}$$

where β_1 is a weighted average of β_{11} and β_{21}. Note that Gupta and Mukerjee (1996) take β_1 as the arithmetic mean of β_{11} and β_{21}. However, with this definition of β_1, the contrasts θ_p, θ_1 and θ_1' in $\tau_1, \ldots, \ldots \tau_v$ are not mutually orthogonal *unless* $m_1 = m_2$. Chai, Das and Dey (2001) suggested that β_1 may be taken as

$$\beta_1 = \frac{\alpha_1\beta_{11} + \alpha_2\beta_{21}}{\alpha_1 + \alpha_2},$$

where for $i = 1, 2$, $\alpha_i = m_i(m_i^2 - 1)$. With the modified definition of β_1 as above, the three contrasts $\theta_p, \theta_1, \theta_1'$ are indeed mutually orthogonal for all m_1, m_2.

The contrasts $\theta_p, \theta_1, \theta_1'$ have natural interpretations. Thus, θ_p is a contrast between the two preparations (and hence, called the *preparation contrast*) while θ_1 and θ_1' are the *combined regression* and *parallelism contrasts*, respectively.

The three contrasts θ_p, θ_1 and θ_1' can be explicitly written as

$$\begin{aligned}
\theta_p &= (m_1^{-1}\mathbf{1}'_{m_1}, -m_2^{-1}\mathbf{1}'_{m_2})\boldsymbol{\tau}, \\
\theta_1 &= \delta_1(\boldsymbol{w}'_1, \boldsymbol{w}'_2)\boldsymbol{\tau}, \\
\theta'_1 &= \delta_2(\alpha_2\boldsymbol{w}'_1, -\alpha_1\boldsymbol{w}'_2)\boldsymbol{\tau},
\end{aligned} \tag{5.3.15}$$

where $\delta_1 = 12/\{(\alpha_1+\alpha_2)\log u\}$, $\delta_2 = 12/(\alpha_1\alpha_2 \log u)$ and for $i = 1, 2$, $\boldsymbol{w}_i = (1, 2, \ldots, m_i)' - \frac{1}{2}(m_i + 1)\mathbf{1}_{m_i}$.

In particular, for *symmetric* parallel line assays, i.e., when $m_1 = m_2 = m$ (say), one can define the treatment contrasts as follows:

$$\begin{aligned}
L_p &= (\boldsymbol{e}'_0, -\boldsymbol{e}'_0)\boldsymbol{\tau}, \\
L_j &= (\boldsymbol{e}'_j, \boldsymbol{e}'_j)\boldsymbol{\tau}, 1 \le j \le m-1, \\
L'_j &= (\boldsymbol{e}'_j, -\boldsymbol{e}'_j)\boldsymbol{\tau}, 1 \le j \le m-1,
\end{aligned} \tag{5.3.16}$$

where for $0 \le j \le m-1$, $\boldsymbol{e}_j = (e_j(1), \ldots, e_j(m))'$ and for $1 \le i \le m$,

$$\begin{aligned}
e_0(i) &= 1, \\
e_1(i) &= i - (m+1)/2, \\
e_2(i) &= \left\{i - \frac{m+1}{2}\right\}^2 - \frac{1}{12}(m^2-1), \\
e_{j+1}(i) &= e_j(i)e_1(i) - \frac{j^2(m^2-j^2)}{4(4j^2-1)}e_{j-1}(i), j \ge 2.
\end{aligned} \tag{5.3.17}$$

It is easily seen that the contrasts L_p, L_1 and L'_1 defined in (5.3.16) are proportional to the earlier defined contrasts θ_p, θ_1 and θ'_1 respectively and in view of this, in the context of symmetric parallel line assays we may interpret the contrasts L_p, L_1, L'_1 as preparation, combined regression and parallelism contrasts, respectively. For $j \ge 2$, the contrasts L_j, L'_j have similar interpretation; for example, L_2 is the combined quadratic contrast while L'_2 measures the difference between quadratic regression coefficients for the two preparations. The conditions that ensure the representability of $\xi_1(\cdot)$ and $\xi_2(\cdot)$ by parallel straight lines can now be expressed as

$$L'_1 = 0,\ L_j = 0,\ L'_j = 0,\ 2 \le j \le m-1. \tag{5.3.18}$$

Also, the relative potency can now be expressed as

$$\rho = (d_1/d_2)\exp\left\{-\frac{1}{6}(m^2-1)(\log u)(L_p/L_1)\right\}. \tag{5.3.19}$$

In most practical situations, either from past experience or prior knowledge, there is a reason to believe that the relationship between the response and log dose is linear for both the preparations, but the parallelism of the two lines might be in doubt. Under such a scenario, interest mainly lies in the contrasts L_p, L_1, L_1'. In other situations, often an experimenter may anticipate that $\xi_1(\cdot)$ and $\xi_2(\cdot)$ may be at most quadratic and then, apart from the contrasts L_p, L_1, L_1', the contrasts L_2, L_2' are also of interest. In general, one should attempt to choose a design that keeps all treatment contrasts at least estimable (and hence, testable).

While the above discussion focused on symmetric parallel line assays, similar considerations hold for asymmetric assays in which $m_1 \neq m_2$. For instance, if the experimenter is confident about the linearity of $\xi_1(\cdot)$ and $\xi_2(\cdot)$ but is not so about their parallelism, then the contrasts of interest are $\theta_p, \theta_1, \theta_1'$.

5.3.3 Block Designs for Parallel Line Assays

We now take up the issue of finding incomplete block designs for parallel line assays that allow the estimation of important contrasts with full information (see below for an explanation of the term "full information"). Consider a parallel line assay involving m_1 doses $s_1, \ldots, s_{m_1}$ of the standard preparation and m_2 doses $t_1, \ldots, t_{m_2}$ of the test preparation. These $v = m_1 + m_2$ doses (or, treatments, in the language of conventional incomplete block designs) will always be written in the order $\{s_1, \ldots, s_{m_1}, t_1, \ldots, t_{m_2}\}$. We follow the notations of Chapter 2 and consider the usual fixed effects model (2.2.1) throughout. The following result of Gupta and Mukerjee (1996) (see Exercise 2.23, Chapter 2) is useful in the sequel.

Lemma 5.3.1 *Suppose $\boldsymbol{p}_1'\boldsymbol{\tau}, \ldots, \boldsymbol{p}_u'\boldsymbol{\tau}$ are all estimable under an incomplete block design d and let $P\hat{\boldsymbol{\tau}} = (\boldsymbol{p}_1'\hat{\boldsymbol{\tau}}, \ldots, \boldsymbol{p}_u'\hat{\boldsymbol{\tau}})'$. Then,*
(i) $\mathbb{D}(P\hat{\boldsymbol{\tau}}) - \sigma^2 P R_d^{-1} P' \geq 0$;
(ii) $\mathbb{D}(P\hat{\boldsymbol{\tau}}) = \sigma^2 P R_d^{-1} P'$ if and only if

$$P R_d^{-1} N_d = \mathbf{0}. \tag{5.3.20}$$

If (5.3.20) holds, then $P\hat{\boldsymbol{\tau}} = P R_d^{-1} \boldsymbol{T}$.

The implications of the results in Lemma 5.3.1 are as follows:

(a) $\sigma^2 P R_d^{-1} P'$ is the dispersion matrix of the BLUE of $P\boldsymbol{\tau}$ in a completely randomized design (that is, under an unblocked design) with the same replication numbers and the same error variance as under the design d (recall Remark 2.2.2). Hence, part (i) of the lemma is intuitively obvious since introduction of block effects in the model can possibly only inflate the dispersion matrix (in the sense that the difference between the dispersion matrix under d and that under an unblocked design is n.n.d.).

(b) The condition (5.3.20) is necessary and sufficient for the estimation of $P\boldsymbol{\tau}$ orthogonally to the block effects. Under this condition, the BLUE of $P\boldsymbol{\tau}$ equals $PR_d^{-1}\boldsymbol{T}$ which is the same as under a completely randomized design, a result that is intuitively anticipated.

(c) For a single contrast $\boldsymbol{p}_i'\boldsymbol{\tau}$ which is estimable under d, from (2.2.30) we know that $\mathrm{Var}(\boldsymbol{p}_i'\hat{\boldsymbol{\tau}})_d = \sigma^2\boldsymbol{p}_i'C_d^-\boldsymbol{p}_i$. But, by part (i) of Lemma 5.3.1, we have (taking $u = 1$), $\mathrm{Var}(\boldsymbol{p}_i'\hat{\boldsymbol{\tau}})_d \geq \sigma^2\boldsymbol{p}_i'R_d^{-1}\boldsymbol{p}_i$. Hence, the efficiency factor (as defined in Section 2.5) of $\boldsymbol{p}_i'\boldsymbol{\tau}$ under d is $\boldsymbol{p}_i'R_d^{-1}\boldsymbol{p}_i/\boldsymbol{p}_i'C_d^-\boldsymbol{p}_i$, which is at most unity. The efficiency factor equals unity if and only if $\boldsymbol{p}_i'R_d^{-1}N_d = \mathbf{0}$. In this case, we say that $\boldsymbol{p}_i'\boldsymbol{\tau}$ is being estimated by the design d with full efficiency (or, with full information). In the same spirit, we say that $P\boldsymbol{\tau}$ is estimated with full efficiency (i.e., each component of $P\boldsymbol{\tau}$ is estimated with full efficiency) if and only if (5.3.20) holds.

We are now in a position to determine suitable incomplete block designs for parallel line assays which ensure full information on some or all contrasts of major importance. Construction of such designs and other related issues have been considered among others, by Das and Kulkarni (1966), Kulshreshtha (1969), Kyi Win and Dey (1980), Nigam and Boopathy (1985), Das (1985), Gupta, Nigam and Puri (1987), Gupta (1988) and Gupta and Mukerjee (1990). In what follows, we present a selection of such results.

(a) *Designs for Symmetric Assays.*

In the context of symmetric parallel line assays, let there be $m \geq 2$ doses of each of the preparations and suppose it is desired to obtain an incomplete block design involving $v = 2m$ doses (treatments), b blocks of size $k(< v)$ such that each of the v doses is replicated r times. Clearly, $bk/v (= r)$ then must be an integer.

As emphasized earlier, for symmetric parallel line assays, the three major contrasts of importance are the preparation (L_p), the combined regression (L_1) and the parallelism (L_1') contrasts. It is therefore desirable that under the chosen design, these contrasts be estimated with full information. Such designs have been called L-designs by Gupta and Mukerjee (1990) and henceforth, we follow this terminology. The incidence matrix N_d of a design d for symmetric parallel line assay may be partitioned as $N_d = (N'_{1d}, N'_{2d})'$, where N_{1d} (respectively, N_{2d}) is the $m \times b$ incidence matrix for the m doses of the standard (respectively, test) preparation. Since in d, each block is of size k and each treatment is replicated r times, we have

$$\begin{aligned} \mathbf{1}'_m N_{1d} + \mathbf{1}'_m N_{2d} &= k\mathbf{1}'_b, \\ N_{1d}\mathbf{1}_b = N_{2d}\mathbf{1}_b &= r\mathbf{1}_m. \end{aligned} \tag{5.3.21}$$

In view of (5.3.16), taking

$$P = \begin{bmatrix} e'_0 & -e'_0 \\ e'_1 & e'_1 \\ e'_1 & -e'_1 \end{bmatrix},$$

remembering that R_d, the diagonal matrix of replication numbers equals rI_v and invoking Lemma 5.3.1, it is seen that a design is an L-design if and only if

$$\begin{bmatrix} e'_0 & -e'_0 \\ e'_1 & e'_1 \\ e'_1 & -e'_1 \end{bmatrix} \begin{bmatrix} N_{1d} \\ N_{2d} \end{bmatrix} = \mathbf{0}.$$

Recalling that $\mathbf{e}_0 = \mathbf{1}_m$ and $\mathbf{e}_1 = (1, 2, \ldots, m)' - \frac{1}{2}(m+1)\mathbf{1}_m$, a characterization of L-designs is provided by the conditions

$$\mathbf{1}'_m N_{1d} = \mathbf{1}'_m N_{2d} = \frac{k}{2}\mathbf{1}'_b, \;\; \mathbf{e}'_1 N_{1d} = \mathbf{e}'_1 N_{2d} = \mathbf{0}. \tag{5.3.22}$$

The above conditions were first obtained by Kyi Win and Dey (1980). The following result is an immediate consequence of (5.3.22).

Lemma 5.3.2 *Given m, b and k, a necessary condition for the existence of an L-design is that $k \equiv 0 \pmod 2$.*

The problem of construction of L-designs is taken up next. This problem has been completely solved for even values of m by Gupta and Mukerjee (1990) and their result is given below.

Theorem 5.3.1 *Let m, the number of doses of each preparation, be an even integer and $k < v(= 2m)$. Then an L-design with parameters v, b, r and k exists if and only if $vr = bk$ and $k \equiv 0 \pmod 4$.*

Proof. Let $m = 2u$ and in view of Lemma 5.3.2, let $k = 2w$. To prove the necessity, it suffices to show that w is even. Let a typical column of N_{1d} be $(z_1, \dots, z_{2u})'$. By invoking the definitions of e_0 and e_1, it follows that

$$\sum_{j=1}^{2u} z_j = w, \quad \sum_{j=1}^{2u} z_j(2j - 2u - 1) = 0.$$

Hence

$$\sum_{j=1}^{2u} z_j(j - u) = \frac{1}{2}w.$$

Since z_j's are nonnegative integers, the necessity follows.

The sufficiency is proved by actual construction of the designs. Let the conditions of the theorem hold, i.e., $vr = bk$ and $k \equiv 0 \pmod 4$. Since $v = 4u$, we have $ur = b(k/4)$. Thus, one can always construct an incomplete block design involving u treatments and b blocks such that each block has size $k/4$ and each treatment is replicated r times. Let M_{1d} and M_{2d} be the $u \times b$ incidence matrices of two such designs, which may or may not be distinct. For a positive integer n, let I_n^* be the $n \times n$ permutation matrix given by

$$I_n^* = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & & & & \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then the incidence matrix of the required L-design is given by (see Gupta, (1989))

$$N_d = \begin{bmatrix} I_u & \mathbf{0} \\ I_u^* & \mathbf{0} \\ \mathbf{0} & I_u^* \\ \mathbf{0} & I_u \end{bmatrix} \begin{bmatrix} M_{1d} \\ M_{2d} \end{bmatrix}.$$

Thus, the sufficiency is established. □

Note that the L-design constructed above is connected if $M = \begin{bmatrix} M_{1d} \\ M_{2d} \end{bmatrix}$ is the incidence matrix of a connected incomplete block design. We illustrate the above construction via the following example.

Example 5.3.1 Suppose it is desired to construct an L-design with $m = 4$ doses of each preparation in $b = 4$ blocks, each of size $k = 4$.

Then, $v = 8, r = 2$ and $u = 2$. Take the matrices M_{1d} and M_{2d} as

$$M_{1d} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad M_{2d} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

The incidence matrix of the required L-design can now be obtained following the construction method given in the proof of Theorem 5.3.1 as follows:

$$N_d = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

This design can easily be checked to be connected.

For odd values of m, we first consider the situation $k = 4$. Following Nigam and Boopathy (1985), one has the following result.

Theorem 5.3.2 *Let $m \geq 3$ be odd. Then an L-design with parameters $v(= 2m), b, r, k = 4$ exists if and only if $vr = 4b$.*

Proof. The necessity is obvious. For proving the sufficiency, let $m = 2u + 1$. Then, $v = 4u + 2, k = 4, b = (2u + 1)s, r = 2s$ where s is a positive integer. Let

$$N_{1d} = \begin{bmatrix} I_u & 0 & I_u^* \\ 0 & 2 & 0 \\ I_u^* & 0 & I_u \end{bmatrix}$$

be a square matrix of order $2u + 1$. Also, let N_{2d} be another matrix whose $(i - 1)$th column equals the ith column of N_{1d}, $2 \leq i \leq 2u + 1$. The required L-design can now be constructed by taking s copies of the design with incidence matrix $\begin{bmatrix} N_{1d} \\ N_{2d} \end{bmatrix}$. This design can be seen to be connected. $\square$

Remark 5.3.1 The designs constructed in Theorems 5.3.1 and 5.3.2 allow the estimation of the contrasts L_j and L'_j with full efficiency when $j \geq 1$ is odd.

For odd m and block size k not necessarily equal to 4, Gupta and Mukerjee (1990) obtained a necessary and sufficient condition for the existence of an L-design for odd $m \leq 15$. Though this range of m suffices for all practical purposes, a general solution to the problem of finding L-designs for every odd m is not yet available. The result of Gupta and Mukerjee (1990) is stated below.

Theorem 5.3.3 *Let $m \geq 3$ be odd, $3 \leq m \leq 15$ and $2 < k < v(= 2m)$. Then an L-design with parameters v, b, r and k exists if and only if $vr = bk$ and k is even.*

Gupta and Mukerjee (1990) tabulated L-designs over the range $3 \leq m \leq 15$ for all parameter values given by Theorem 5.3.3. We give below one such design.

Example 5.3.2 Let $m = 5, k = 6$. Then, $v = 10$ and $vr = bk$ yields $b/r = 5/3$. This implies that $b = 5s$ and $r = 3s$ for some positive integer s. Consider the simplest case of $s = 1$. Let $(z_1, \ldots, z_5)'$ be a column of N_{1d}. Then, the nonnegative integers $z_1, \ldots, z_5$ must satisfy

$$\sum_{j=1}^{5} z_j = 3, \quad -2z_1 - z_2 + z_4 + 2z_5 = 0. \tag{5.3.23}$$

The only solutions of (5.3.23) are

$$\begin{aligned} &\boldsymbol{\alpha}_1 = (0,0,3,0,0)', \quad \boldsymbol{\alpha}_2 = (0,1,1,1,0)', \\ &\boldsymbol{\alpha}_3 = (1,0,1,0,1)', \quad \boldsymbol{\alpha}_4 = (0,2,0,0,1)', \\ &\boldsymbol{\alpha}_5 = (1,0,0,2,0)'. \end{aligned}$$

Now let $\boldsymbol{\alpha}_i$ occur u_i times as a column in N_{1d}. Then, since each row sum of N_{1d} must equal $r(= 3)$, we have $u_3 + u_5 = u_2 + 2u_4 = 3u_1 + u_2 + u_3 = u_2 + 2u_5 = u_3 + u_4 = 3$. A solution of these equations is $u_1 = 0, u_2 = 1, u_3 = 2, u_4 = 1 = u_5$. Thus,

$$N_{1d} = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

The incidence matrix of a connected L-design with parameters $v = 10, b = 5, k = 6, r = 3$ is given by $N_d = \begin{bmatrix} N_{1d} \\ N_{1d} \end{bmatrix}$. An L-design with $3s$ replications is obtained by taking s copies of the above design.

For given v, b, k, an L-design may not exist even if k is even. In such situations, one might wish to find designs that retain full information on any two of the three major contrasts, L_p, L_1 and L_1'. Attempts to find such designs have been made in the literature and to that end, the following results have been reported.

Theorem 5.3.4 *Let $vr = bk$. Then an equireplicate design, with parameters $v(= 2m), b, r, k$ retaining full information on L_p and any one of L_1 and L_1' exists if and only if k is even.*

Proof. The necessity is obvious by noting that if such a design exists then the matrices N_{1d}, N_{2d} corresponding to the design must satisfy the first condition in (5.3.22). The sufficiency can be proved by actual construction, as given by Das and Kulkarni (1966). Since $mr = b(k/2)$, if k is even then one can always construct a design involving m treatments and b blocks such that each block has size $k/2$, each treatment being replicated r times. Let N_{1d} be the $m \times b$ incidence matrix of such a design. Define $N_{2d} = I_m^* N_{1d}$, that is, N_{2d} is obtained by permuting the rows of N_{1d} in the reverse order. Then, it can be seen that the design with incidence matrix $N^{(1)} = \begin{bmatrix} N_{1d} \\ N_{2d} \end{bmatrix}$ retains full information on L_p and L_1. Similarly, the design with incidence matrix $N^{(2)} = \begin{bmatrix} N_{1d} \\ N_{1d} \end{bmatrix}$ retains full information on L_p and L_1'. □

Remark 5.3.2 The designs with incidence matrix $N^{(i)}$, $i = 1, 2$ are connected provided N_{1d} is the incidence matrix of a connected incomplete block design. Also, the design with incidence matrix $N^{(1)}$ retains full information on L_j for every odd j and on L_j' for every even j. Similarly, the design with incidence matrix $N^{(2)}$ also ensures full efficiency on L_j' for every j.

Example 5.3.3 Let $m = 4, b = 4, r = 3, k = 6$. For these parameters, an L-design is not available. Let us take N_{1d} of Theorem 5.3.4 as the incidence matrix of a BIB design with $m = 4$ treatments in blocks of size $k/2 = 3$, which is given below:

$$N_{1d} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

The incidence matrix $N^{(1)}$ is then given by

$$N^{(1)} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

The design with incidence matrix $N^{(1)}$ retains full information on L_p, L_1, L_3, L_2' and has an efficiency factor of 8/9 for the contrasts L_1', L_3', L_2.

Remark 5.3.3 Chai (2002) has shown that for a symmetric parallel line assay with m doses of each preparation, an L-design exists if and only if $\frac{1}{2}k(m+1) \equiv 0 \pmod 2$. Thus, one cannot construct an L-design if either of the following two conditions hold:
(i) k is odd and m is even;
(ii) $k \equiv 2 \pmod 4$ and m is even.

For case (ii) above, Chai and Das (2001) obtained designs that ensure the estimability of L_p and L_1 with full efficiency; these designs were called nearly L-designs. When the block size is odd, Chai, Das and Dey (2003) obtained incomplete block designs for symmetric parallel line assays that are highly efficient for the estimation of the contrasts L_p, L_1 and L_1'. The original sources may be consulted for details on these.

In the context of symmetric parallel line assays, some times it might be necessary to find designs that are capable of providing full information on the second order contrasts L_2 and L_2' in addition to L_p, L_1, L_1'. Such designs have been studied by Mukerjee and Gupta (1991a) who call these as Q-designs. We discuss some basic issues concerning Q-designs now.

Consider an arrangement of $v(= 2m)$ treatments (or, doses) in b blocks each of size $k(< v)$ such that each treatment is replicated r times. The incidence matrix N_d of the design is partitioned as before as $N_d = (N_{1d}', N_{2d}')'$, where N_{1d}, N_{2d} are as defined earlier. The design under consideration is a Q-design if and only if

$$\mathbf{1}_m' N_{id} = (\frac{1}{2}k)\mathbf{1}_b', \; e_1' N_{id} = \mathbf{0}, \; e_2' N_{id} = \mathbf{0}, \; i = 1, 2, \qquad (5.3.24)$$

where e_1 is as defined earlier,

$$e_2 = f_2 - \frac{1}{12}(m^2 - 1)\mathbf{1}_m$$

and f_2 is an $m \times 1$ vector with its jth element equal to $(j-(m+1)/2)^2, 1 \leq j \leq m$. Clearly, as in the case of an L-design, a necessary condition for the existence of a Q-design is that k is even. Also, if $(z_1, \ldots, z_m)'$ is a typical column of N_{1d} or N_{2d}, by (5.3.24), we must have

$$\begin{aligned}
\sum_{j=1}^{m} z_j &= \frac{1}{2}k, \\
\sum_{j=1}^{m} j z_j &= \frac{1}{4}k(m+1), \\
\sum_{j=1}^{m} \left\{ j - \frac{1}{2}(m+1) \right\}^2 z_j &= \frac{1}{24}k(m^2 - 1).
\end{aligned} \tag{5.3.25}$$

Based on the above facts, Mukerjee and Gupta (1991a) suggested a construction procedure involving the following steps:

Step 1. For given v, k, r, search all possible nonnegative integral solutions of (5.3.25) for $(z_1, \ldots, z_m)'$. If no such solution exists, then a Q-design with the given parameters also does not exist. Otherwise, let $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_s$ be the possible solutions.

Step 2. Find nonnegative integers $u_1, \ldots, u_s$ such that $\sum_{i=1}^{s} u_i \boldsymbol{\alpha}_i = r\mathbf{1}_m$. If no such u_i's exist, then a Q-design with the given parameters does not exist. Otherwise, construct N_{1d} with columns $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_s$ such that $\boldsymbol{\alpha}_i$ is repeated u_i times, $1 \leq i \leq s$. Finally, take $N_{2d} = N_{1d}$. The incidence matrix so obtained represents a Q-design with the given parameters.

The above approach was adopted by Mukerjee and Gupta (1991a) to obtain several Q-designs in the parametric range $v \leq 24, k < v$. They also noted that the resulting design is always connected for all $r > 1$.

Example 5.3.4 We construct a Q-design with parameters $v = 14, b = 7, r = 6, k = 12$. The possible nonnegative integral solutions of (5.3.25) are

$$\boldsymbol{\alpha}_1 = (0,3,0,0,0,3,0)', \ \boldsymbol{\alpha}_2 = (1,1,0,1,2,0,1)' \ \boldsymbol{\alpha}_3 = (1,0,2,1,0,1,1)'.$$

Since $\boldsymbol{\alpha}_1 + 3\boldsymbol{\alpha}_2 + 3\boldsymbol{\alpha}_3 = 6\mathbf{1}_m$, a Q-design can be constructed whose incidence matrix is $N_d = (N'_{1d}, N'_{1d})'$ where

$$N_{1d} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

(b) *Designs for Asymmetric Assays*

We now take up asymmetric assays where $m_1 \neq m_2$ and seek designs that allow the estimation of the contrasts θ_p, θ_1 and θ'_1 with full information. Such designs may be called θ-designs. Recall that

$$\begin{aligned}
\theta_p &= m_1{}^{-1} \sum_{i=1}^{m_1} \tau_i - m_2{}^{-1} \sum_{i=1}^{m_2} \tau_{m_1+i}, \\
\theta_1 &= \frac{6}{\log u} \left[\frac{1}{m_1(m_1^2-1)} \sum_{i=1}^{m_1} \left\{ i - \frac{1}{2}(m_1+1) \right\} \tau_i \right. \\
&\quad \left. + \frac{1}{m_2(m_2^2-1)} \sum_{i=1}^{m_2} \left\{ i - \frac{1}{2}(m_2+1) \right\} \tau_{m_1+i} \right], \\
\theta'_1 &= \frac{6}{\log u} \left[\frac{1}{m_1(m_1^2-1)} \sum_{i=1}^{m_1} \left\{ i - \frac{1}{2}(m_1+1) \right\} \tau_i \right. \\
&\quad \left. - \frac{1}{m_2(m_2^2-1)} \sum_{i=1}^{m_2} \left\{ i - \frac{1}{2}(m_2+1) \right\} \tau_{m_1+i} \right]. \qquad (5.3.26)
\end{aligned}$$

Suppose an assay involving m_1 doses of the standard preparation and m_2 doses of the test preparation is to be conducted in an incomplete block design with b blocks of size $k(< v = m_1 + m_2)$ such that each dose is replicated r times. Let the incidence matrix of the design be N_d where as before, N_d is partitioned as $N_d = (N'_{1d}, N'_{2d})'$, the rows of N_{1d} (respectively, N_{2d}) representing the doses of standard (respectively, test) preparation. Then,

$$\mathbf{1}'_{m_1} N_{1d} + \mathbf{1}'_{m_2} N_{2d} = k \mathbf{1}'_b. \qquad (5.3.27)$$

Also, it can be seen that the design under consideration is a θ-design if and only if, for $i = 1, 2$,

$$\mathbf{1}'_{m_i} N_{id} = (km_i/v) \mathbf{1}'_b,$$

$$\left\{\boldsymbol{f}_i - \frac{1}{2}(m_i+1)\mathbf{1}_{m_i}\right\}' N_{id} = \mathbf{0}, \tag{5.3.28}$$

where $\boldsymbol{f}_i = (1,2,\ldots,m_i)'$, $i = 1,2$. The conditions (5.3.28) are due to Kyi Win and Dey (1980). The construction of θ-designs was also considered by Kyi Win and Dey (1980) who presented a short table of such designs. One such design is considered in the following example.

Example 5.3.5 Let $m_1 = 3, m_2 = 6, b = 3, r = 2, k = 6$. Kyi Win and Dey (1980) showed that the design with incidence matrix given by

$$N_d = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

is a θ-design with the given parameters.

If both m_1 and m_2 are even, then a complete solution to the problem of constructing θ-designs can be obtained as given by the following result.

Theorem 5.3.5 *Let $m_1 = 2u_1$ and $m_2 = 2u_2$ be both even and $vr = bk$. Then a θ-design with parameters $v(= m_1 + m_2), b, r$ and k exists if and only if k is even and ku_1/v is an integer.*

Proof. Suppose a θ-design with the given parameters and associated incidence matrices N_{1d}, N_{2d} exists and let $(z_1, \ldots, z_{2u_1})'$ be a typical column of N_{1d}. Then, by (5.3.28),

$$\sum_{j=1}^{2u_1} z_j(j-u_1) = \frac{1}{2}\sum_{j=1}^{2u_1} z_j = ku_1/v. \tag{5.3.29}$$

From (5.3.29), ku_1/v must be an integer. Then by (5.3.29), the column sums of both N_{1d} must be even. Similarly, the column sums of N_{2d} are also even. Hence by (5.3.27), k must be even. This proves the necessity.

The sufficiency can be proved by actual construction. Let k be even and ku_1/v be an integer. Then, $ku_2/v = k/2 - ku_1/v$ is also an integer. Also, for $i = 1,2$, $u_i r = b(ku_i/v)$. Hence for $i = 1,2$, one can always construct an incomplete block design involving u_i treatments and b blocks such that every treatment is replicated r times and each block is of size ku_i/v. For $i = 1, 2$, let M_{id} be the $u_i \times b$ incidence matrix of such an incomplete block design. Then the incidence matrix of the required θ-design d is given by

$$N_d = \begin{bmatrix} I_{u_1} & \mathbf{0} \\ I^*_{u_1} & \mathbf{0} \\ \mathbf{0} & I^*_{u_2} \\ \mathbf{0} & I_{u_2} \end{bmatrix} \begin{bmatrix} M_{1d} \\ M_{2d} \end{bmatrix}.$$

The above design is connected if $M = (M'_{1d}, M'_{2d})'$ is the incidence matrix of a connected design. □

Example 5.3.6 Let $m_1 = 4, m_2 = 8, b = 4, r = 2, k = 6$. Then the conditions of Theorem 5.3.5 are satisfied. One can take the matrices $M_{id}, i = 1, 2$ as

$$M_{1d} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \quad M_{2d} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

Given the above matrices, one can now get the incidence matrix of the required θ-design as indicated in the proof of Theorem 5.3.5. This design was also reported by Kyi Win and Dey (1980).

Das and Saha (1986) used affine resolvable incomplete block designs and the C-designs considered in Chapter 4 for the construction of θ-designs. Through the use of C-designs, these authors also presented non-equireplicate designs which estimate the three contrasts θ_p, θ_1 and θ'_1 with full information.

5.4 Designs for Test-Control Comparisons

In practice one often encounters the following situation: v new (or, *test*) treatments are available and an existing treatment (called a *control*) is to be eventually replaced by one of the test treatments. For instance, in pharmaceutical studies, new drugs are the test treatments and a placebo or an existing drug is the control treatment. Similarly, in experiments for the assessment of crop varieties, an existing variety may be taken as control and newly developed varieties may be considered as test treatments.

The interest in such experiments is to infer about the contrasts between the control and each of the test treatments. Because of the special nature of the treatment contrasts of interest, a conventional design might not be a good choice for the problem under consideration. To elaborate on this point, consider an experiment involving $v = 7$ test treatments and a control. Let the test treatments be labeled as $1, 2, \dots, 7$ and the control as 0. Suppose d_1 is the BIB design given in Example 3.4.2 and d_2 is the design obtained by taking two copies of the following design:

$$\begin{array}{cccc} (0,1,2,4) & (0,2,3,5) & (0,3,4,6) & (0,4,5,7) \\ (0,1,5,6) & (0,2,6,7) & (0,1,3,7) & \end{array}.$$

Note that both d_1 and d_2 involve 14 blocks of size 4 each and d_2 is *not* a BIB design. If τ_0 is the effect of the control treatment and τ_i, that of the ith test treatment ($1 \le i \le 7$), then the variance of the BLUE of the contrast $\tau_i - \tau_0$ under d_1 and d_2 can be seen to be

$$\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_0)_{d_1} = 0.3334\sigma^2, \ \mathrm{Var}(\hat{\tau}_i - \hat{\tau}_0)_{d_2} = 0.2667\sigma^2, \ 1 \le i \le 7.$$

Thus, in this case, the BIB design d_1 is inferior to the design d_2 for estimating the contrasts $\tau_i - \tau_0$.

Most of the recent work carried out in the area of determining designs for test-control comparisons centers around finding optimal designs for such experiments. This aspect will be covered in the next chapter. In this section, we present the different types of incomplete block designs available for the problem.

Throughout this section, the label 0 is reserved for the control while the test treatment labels are $1, 2, \dots, v$. Suppose an incomplete block design d involving $v + 1$ treatments (v test and a single control treatment), b blocks each of size k and incidence matrix $N_d = (n_{dij})$ is to be used for the experiment. For the design d, let r_{di}, $0 \le i \le v$, denote the replication of the ith treatment and let $\lambda_{dii'} = \sum_{j=1}^{b} n_{dij} n_{di'j}$, $i, i' = 0, 1, \dots, v, i \ne i'$.

Cox (1958, p. 238) recommended the use of BIB designs for such an experiment in the following manner: Start with a BIB design d_1 involving v test treatments, b blocks each of size $k - t$, r replications and concurrence parameter λ and then augment each block of d_1 by t replications of the control; here t is a positive integer. Such designs were called *reinforced BIB designs* by Das (1958). The original rationale behind this kind of designs was that the test treatments are balanced and each of the test treatments is balanced with respect to the control.

Pearce (1960) suggested the use of a class of designs, called *supplemented balanced designs* for the control-test comparison experiments. Such designs were introduced by Hoblyn, Pearce and Freeman (1954) in another context. Supplemented balanced designs include the reinforced BIB designs as a special case. A formal definition of these designs follows.

Definition 5.4.1 *An incomplete block design d involving v test treatments, a control and b blocks each of size k is called a design with supplemented balance with 0 as the supplemented treatment if there exist nonnegative integers λ_{d0} and λ_{d1}, such that*

$$\begin{aligned} \lambda_{dii'} &= \lambda_{d1}, \text{ for } i, i' = 1, \ldots, v, i \neq i', \\ \lambda_{d0i} &= \lambda_{d0}, \text{ for } i = 1, \ldots, v. \end{aligned} \tag{5.4.1}$$

Example 5.4.1 Apparently, the first design with supplemented balance is one that was used in an experiment involving strawberry plants at the East Malling Research Station (Pearce, 1953). The test treatments were four herbicides, labeled 1,2,3,4 which were compared with a control (no herbicide), labeled 0 in four blocks of size seven each. The full design is shown below.

$$\begin{array}{ll} \text{Block 1}: & (0,0,1,2,3,4,1) \\ \text{Block 2}: & (0,0,1,2,3,4,2) \\ \text{Block 3}: & (0,0,1,2,3,4,3) \\ \text{Block 4}: & (0,0,1,2,3,4,4) \end{array}$$

Clearly, for this design $\lambda_{d0} = 10, \lambda_{d1} = 6$.

Note that for a reinforced BIB design of Das (1958), $\lambda_{d0} = rt$. The analysis of designs with supplemented balance was worked out by Pearce (1960) who also observed that the variance of the BLUE of an elementary treatment contrast involving a pair of test treatments is a constant and similarly, the variance of the BLUE of an elementary treatment contrast involving the control and a test treatment is a constant.

An important class of designs suitable for test-control treatment comparisons is the class of *balanced test treatment incomplete block* (BTIB) designs, introduced by Bechhofer and Tamhane (1981).

Definition 5.4.2 *An incomplete block design d involving v test treatments, a control and b blocks each of size k is called a BTIB design if (i) $2 \leq k < v$ and (ii) the conditions in (5.4.1) hold.*

Clearly, the BTIB designs are also designs with supplemented balance. Bechhofer and Tamhane (1981) while considering the problem of constructing simultaneous confidence intervals for the treatment-control contrasts, rediscovered supplemented balanced designs and called these BTIB designs. In subsequent years, the term "BTIB designs" has been adopted by most authors because the work of Bechhofer and Tamhane (1981) inspired much of the subsequent research on finding optimal and efficient designs for test-control comparisons. It can be easily seen that under a BTIB design d,

$$\mathrm{Var}(\hat{\tau}_i - \hat{\tau}_0)_d = \sigma^2 k \frac{\lambda_{d0} + \lambda_{d1}}{\lambda_{d0}(\lambda_{d0} + v\lambda_{d1})}.$$

A special type of useful BTIB designs, called BTIB $(v, b, k; t, s)$ are defined next (see Stufken (1987)).

Definition 5.4.3 *An incomplete block design d involving v test treatments, a control, b blocks each of size k and incidence matrix $N_d = (n_{dij})$, $0 \le i \le v, 1 \le j \le b$, is called a BTIB $(v, b, k; t, s)$ if it is a BTIB design, and with $0 \le t \le k-1, 0 \le s \le b$, the following holds:*

$$n_{dij} \in \{0, 1\}, \textit{ for all } 1 \le i \le v, \ 1 \le j \le b$$

and

$$n_{d01} = \cdots = n_{d0s} = t+1, \ n_{d0(s+1)} = \cdots = n_{d0b} = t.$$

For a BTIB $(v, b, k; t, s)$ design d,

$$\begin{aligned} r_{d0} &= bt + s, \\ \lambda_{d0} &= v^{-1}\{bt(k-t) + s(k-2t-1)\}, \\ \lambda_{d1} &= \{v(v-1)\}^{-1}(b(k-t) - 2s)(k-t-1). \end{aligned}$$

We give below some examples of BTIB $(v, b, k; t, s)$ designs (*cf.* Majumdar (1996)).

Example 5.4.2 (i) A BTIB $(7, 7, 4; 1, 0)$ is shown below (which is the same as d_2 given earlier), where the parentheses include the blocks.

$$(0,1,2,4); (0,2,3,5); (0,3,4,6); (0,4,5,7); (0,5,6,1); (0,6,7,2); (0,7,1,3).$$

Note that that the above BTIB design is obtained by simply augmenting each block of a BIB design with parameters $v = 7 = b, r = 3 = k, \lambda = 1$ and is thus a reinforced BIB design.

(ii) A BTIB $(6,18,5;1,6)$ is shown below:

$$(0,0,1,2,3);(0,0,1,4,5);(0,0,1,2,6);(0,0,2,4,5);(0,0,3,4,6);$$

$$(0,0,3,5,6);(0,1,2,3,4);(0,1,2,5,6);(0,1,2,3,5);(0,1,2,4,6);$$

$$(0,1,3,4,5);(0,1,4,5,6);(0,1,3,5,6);(0,1,3,4,6);(0,2,3,4,5);$$

$$(0,2,3,4,6);(0,2,3,5,6);(0,2,4,5,6).$$

As the following example shows, not all BTIB designs are BTIB $(v,b,k;t,s)$ designs.

Example 5.4.3 The following design is a BTIB design with $v=4$, $b=12, k=4$ but not a BTIB $(4,12,4;t,s)$:

$$(0,0,0,1);(0,0,0,2);(0,0,0,3);(0,0,0,4);(0,0,1,2);(0,0,1,3);$$

$$(0,0,1,4);(0,0,2,3);(0,0,2,4);(0,0,3,4);(1,2,3,4);(1,2,3,4).$$

It can be seen that the structure of a BTIB $(v,b,k;t,s)$ can be of two types. If $s=0$, then the design is called an R-type design (or, *rectangular* type design) whereas if $s>0$, then it is called an S-type (or, *step* type design). This terminology is due to Hedayat and Majumdar (1984). An R-type design with columns as blocks can be visualized as a $k \times b$ array, given by

$$d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \tag{5.4.2}$$

where d_1 is a $t \times b$ array of controls while d_2 is a $(k-t) \times b$ array in the test treatments only. It follows then that d_2 must be a BIB design with v treatments, b blocks of size $k-t$ each, r replications and pairwise concurrence parameter λ. Therefore, the construction problem of R-type designs reduces to that of a suitable BIB design (recall Example 5.4.2 (i)).

An S-type design can be visualized as the following $k \times b$ array:

$$d = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}, \tag{5.4.3}$$

where d_{11} is a $(t+1) \times s$ array of controls, d_{12} is a $t \times (b-s)$ array of controls, d_{21} is a $(k-t-1) \times s$ array of test treatments and d_{22} is a $(k-t) \times (b-s)$ array of test treatments.

The following result is due to Hedayat and Majumdar (1984).

Theorem 5.4.1 *(i) For the existence of a* BTIB $(v, b, k; t, s)$ *where* $r_0 = bt + s$, *the following conditions are necessary:*

$$v^{-1}(b(k-t)-s) = v^{-1}(bk - r_0) \text{ is an integer, say } q_1. \tag{5.4.4}$$

$$v^{-1}s(k-t-1) \text{ is an integer, say } q_2. \tag{5.4.5}$$

$$(v-1)^{-1}\{q_2(k-t-2)+(q_1-q_2)(k-t-1)\} \text{ is an integer.} \tag{5.4.6}$$

(ii) For an R-type design, it is necessary that $b \geq v$ and for an S-type design, it is necessary that $b \geq v+1$.

The construction of R-type BTIB designs does not pose any special problems because as noted earlier, these BTIB designs can be constructed easily from a BIB design. The construction of BTIB designs of S-type is not that simple. The construction of such designs for $k = 2$ was considered by Bechhofer and Tamhane (1983) and Notz and Tamhane (1983) gave a complete solution for $k = 3, 3 \leq v \leq 10$. Hedayat and Majumdar (1984) gave an elaborate table of optimal designs in the parametric range $2 \leq k \leq 8, k \leq v \leq 30$ and $v \leq b \leq 50$ which contains several S-type designs. More on the construction of S-type designs can be found in Cheng, Majumdar, Stufken and Ture (1988) and Ture (1982).

A generalization of BTIB designs was proposed and studied by Jacroux (1987), who called these *group divisible treatment design* (GDTD). These were defined by Jacroux (1987) as follows.

Definition 5.4.4 *An incomplete block design d with $v = mn$ test treatments, a control, b blocks each of size k is called a GDTD design with parameters $m, n, \lambda_0, \lambda_1, \lambda_2$, if the treatments $1, 2, \ldots, v$ can be divided into m disjoint sets $S_1, \ldots, S_m$, of n treatments each such that there are nonnegative constants $\lambda_0, \lambda_1, \lambda_2$, satisfying the following conditions:*

$$\begin{aligned}
\lambda_{d0i} &= \lambda_0, \text{ for } 1 \leq i \leq v, \\
\lambda_{dii'} &= \lambda_1, \text{ for } i, i' \in S_c,\ i \neq i', \\
\lambda_{dii'} &= \lambda_2, \text{ for } i \in S_c, i' \in S_e,\ c, e \in \{1, 2, \ldots, m\}, c \neq e.
\end{aligned}$$

For a GDTD design, the variance of the BLUE of elementary contrasts between a test treatment and the control treatment, i.e., $\text{Var}(\hat{\tau}_i - \hat{\tau}_0)$, $1 \leq i \leq v$ are all equal. However, $\text{Var}(\hat{\tau}_i - \hat{\tau}_i')$, $1 \leq i, i' \leq v, i \neq i'$ can take at most two values depending on whether i, i' belong to the same set or not. Analogous to the definition of a BTIB $(v, b, k; t, s)$ design, one can define a GDTD $(v, b, k; t, s)$ design as follows.

Definition 5.4.5 *An incomplete block design d involving v test treatments, a control, b blocks each of size k and incidence matrix $N_d = (n_{dij})$, $0 \leq i \leq v, 1 \leq j \leq b$, is called a GDTD $(v, b, k; t, s)$ if it is a GDTD design, and with $0 \leq t \leq k-1$, $0 \leq s \leq b-1$, the following holds:*

$$n_{dij} \in \{0,1\}, \text{ for all } 1 \leq i \leq v, \ 1 \leq j \leq b$$

and

$$n_{d01} = \cdots = n_{d0s} = t+1, \ n_{d0(s+1)} = \cdots = n_{d0b} = t.$$

It can be easily seen that a GDTD $(v, b, k; t, 0)$ can be obtained by augmenting each block of a group divisible incomplete block design with block size $k-t$ by t replications of the control. For some more methods of construction of GDTD designs we refer to Jacroux (1987) and Stufken (1991).

An obvious extension of the problem of comparing several test treatments with a single control is the problem of comparing a set of test treatments with several controls. The problem of finding suitable designs for this situation has received some attention in the literature and we refer to Majumdar (1986) and Christof (1987) for more details. For some other related problems and discussion on these, one might refer to the authoritative review articles by Hedayat, Jacroux and Majumdar (1988) and Majumdar (1996), where more references can be found.

5.5 Designs for Diallel Crosses

The diallel cross is a type of mating design used in plant breeding to study the genetic properties of a set of inbred lines. The purpose of such an experiment is to compare lines with respect to their *general combining abilities* (g.c.a.). Apart from inferring on general combining abilities, often an experimenter is also interested in inference on "cross effects" or, *specific combining abilities* (s.c.a.). For genetic interpretation of these parameters, we refer to Griffing (1956) and Hinkelmann (1975).

Experimental (or, environmental) design issues in the context of diallel cross experiments has received considerable attention in the literature; see e.g., Curnow (1963), Hinkelmann (1975), Hinkelmann and Kempthorne (1963), Singh and Hinkelmann (1995) and Gupta, Das and Kageyama (1994).

A common diallel cross experiment involves $v = p(p-1)/2$ crosses of the type $(i \times j), i, j = 1, \ldots p, \ i < j$, where p is the number of inbred lines involved. This is the type IV mating design of Griffing (1956) and we concern ourselves in this section with this type of mating designs only. A diallel cross of this type has also been called a *complete* diallel cross. For notational convenience the cross $(i \times j)$ will sometimes be denoted by the ordered pair (i, j). The number of crosses in such a mating design increases rapidly with increase in the number of lines, p ; for $p = 5$, the number of crosses is 10 while for $p = 10$, the number of crosses increases to 45. If there is heterogeneity in one direction in the experimental material, then adoption of a randomized complete block design with crosses as treatments would result in a large error variance even with moderate number of inbred lines. In order to control the error, one would therefore look for a suitable incomplete block design for diallel crosses. One possibility in this direction is to use available incomplete block designs, for instance a BIB design, for the diallel experiment, treating the treatments of the BIB design as crosses. Such an approach has been followed, for example by Das and Giri (1986) and Ceranka and Mejza (1988). Another approach proposed is to start with an incomplete block design for the usual treatment-block structure, treat the treatments as lines and make all possible pairwise crosses among the lines within a block; see e.g., Ghosh and Divecha (1997) and Sharma (1998). However, subsequent research on optimal incomplete block designs for diallel crosses (see Dey (2002) for a review) has shown that the above approaches are not entirely satisfactory as even a highly efficient conventional incomplete block design when used for a diallel cross experiment may turn out to be rather inefficient. It is thus necessary to devise special techniques for obtaining efficient (or, even optimal) incomplete block designs for diallel cross experiments. In what follows, we describe some attempts in this direction.

In a diallel cross experiment, the v crosses are regarded as treatments. If the fixed effect of the cross (i, j) is denoted by τ_{ij}, then we have the representation

$$\tau_{ij} = \bar{\tau} + g_i + g_j + s_{ij}, \tag{5.5.1}$$

where $\bar{\tau}$ is the mean effect of the treatments (crosses), the $\{g_i\}$ stand for the g.c.a. effects, $\{s_{ij}\}$ denote the s.c.a. effects, and

$$\begin{aligned} g_1 + \cdots + g_p &= 0, & (5.5.2)\\ s_{1i} + \cdots + s_{(i-1)i} + s_{i(i+1)} + s_{ip} &= 0, \ 1 \le i \le p. & (5.5.3) \end{aligned}$$

The g.c.a. effects are the effects of individual lines and the s.c.a. effects are those of crosses. In what follows, we arrange the crosses in the order $(1,2),(1,3),\ldots,(1,p),(2,3),\ldots,(2,p),\ldots,(p-1,p)$. Let $\boldsymbol{g}=(g_1,\ldots,g_p)'$ and $\boldsymbol{\tau}$ and $\boldsymbol{s}$ be $v\times 1$ vectors of elements $\{\tau_{ij}\}$ and $\{s_{ij}\}$ respectively. Following Chai and Mukerjee (1999), the general and specific combining ability effects can be expressed in terms of $\boldsymbol{\tau}$ as detailed below.

Let U be a $p\times v$ matrix with rows indexed by $1,\ldots,p$ and columns by the pairs $(i,j), i,j=1,\ldots,p, i<j$ such that the $[u,(i,j)]$th entry of U is 1 if $u\in(i,j)$ and is zero, otherwise. One can then see that

$$UU'=(p-2)I_p+J_p,\quad (UU')^{-1}=(p-2)^{-1}\left(I_p-\frac{1}{2p-2}J_p\right), \tag{5.5.4}$$

$$U\mathbf{1}_v=(p-1)\mathbf{1}_p,\quad U'\mathbf{1}_p=2\mathbf{1}_v. \tag{5.5.5}$$

We can now express (5.5.1) as

$$\boldsymbol{\tau}=\bar{\tau}\mathbf{1}_v+U'\boldsymbol{g}+\boldsymbol{s}, \tag{5.5.6}$$

where, from (5.5.2) and (5.5.3),

$$\mathbf{1}_p'\boldsymbol{g}=0,\quad U\boldsymbol{s}=\mathbf{0}. \tag{5.5.7}$$

Premultiplying (5.5.6) by U and using (5.5.4), (5.5.5) and (5.5.7), one gets

$$\boldsymbol{g}=H_1\boldsymbol{\tau},\quad \boldsymbol{s}=\boldsymbol{\tau}-\bar{\tau}\mathbf{1}_v-U'\boldsymbol{g}=H_2\boldsymbol{\tau}, \tag{5.5.8}$$

where

$$H_1=(UU')^{-1}U-\frac{1}{2v}J_{pv}=\frac{1}{p-2}\left(U-\frac{2}{p}J_{pv}\right), \tag{5.5.9}$$

and

$$H_2=I_v-U'(UU')^{-1}U=I_v-\frac{1}{p-2}\left(U'U-\frac{2}{p-1}J_v\right). \tag{5.5.10}$$

Since

$$\begin{aligned}
H_1\mathbf{1}_v &= \mathbf{0},\\
H_2\mathbf{1}_v &= \mathbf{0},\\
H_1H_2' &= \mathbf{0},\\
Rank(H_1) &= p-1,\\
Rank(H_2) &= v-p,
\end{aligned} \tag{5.5.11}$$

it follows that $\boldsymbol{g}$ and $\boldsymbol{s}$ represent treatment contrasts carrying $p-1$ and $v-p$ degrees of freedom, respectively, and the contrasts represented by $\boldsymbol{g}$ are orthogonal to those represented by $\boldsymbol{s}$. Note that for $p=3$, $\boldsymbol{s}=\boldsymbol{0}$ and thus, when the s.c.a. effects are included in the model, one has to take $p \geq 4$. The above development due to Chai and Mukerjee (1999) gives a convenient and complete description of the contrasts belonging to the g.c.a and s.c.a. effects.

Barring a few exceptions, most of the work on optimal incomplete block designs for diallel crosses have been derived under a model that includes the block effects and only the g.c.a. effects but no s.c.a. effects. While the optimality aspects of incomplete block designs for diallel crosses are deferred to the next chapter, in what follows we describe some incomplete block designs under a model that has only the g.c.a. effects, apart from the block effects. Suppose d is an incomplete block design used for a diallel cross experiment involving $v=p(p-1)/2$ crosses, where p is the number of lines involved, and b blocks, each of size k. Furthermore, let r_{di} denote the number of times the ith cross appears in d, $1 \leq i \leq p(p-1)/2$, and similarly, let s_{dj} denote the number of times the jth line occurs in d, $1 \leq j \leq p$. It is then easy to see that

$$\sum_{i=1}^{\frac{p(p-1)}{2}} r_{di} = bk, \quad \sum_{j=1}^{p} s_{dj} = 2bk.$$

For the data obtained from d, we postulate the model

$$\boldsymbol{Y} = \mu \mathbf{1}_n + \Delta_1 \boldsymbol{g} + \Delta_2 \boldsymbol{\beta} + \boldsymbol{\epsilon}, \tag{5.5.12}$$

where $\boldsymbol{Y}$ is the $n \times 1$ vector of observed responses, μ is a general mean effect, $\boldsymbol{g}$ and $\boldsymbol{\beta}$ are vectors of p g.c.a. and b block effects respectively, Δ_1, Δ_2 are the corresponding design matrices; i.e., the (t,t')th element of Δ_1 is 1 if the tth observation pertains to the t'th line and is zero, otherwise. Similarly, the (u,u')th element of Δ_2 is 1 if the uth observation comes from the u'th block and is zero, otherwise. $\boldsymbol{\epsilon}$ is the $n \times 1$ vector of random error components, these having a zero mean and constant variance σ^2; n is the total number of experimental units in d. Under the model (5.5.12), the reduced intra-block normal equations for estimating linear functions of the g.c.a. effects, using the design d are given by

$$C_d \boldsymbol{g} = \boldsymbol{Q}, \tag{5.5.13}$$

where

$$C_d = G_d - k^{-1} N_d N_d', \tag{5.5.14}$$

N_d is the $p \times b$ incidence matrix of *lines versus blocks* and $\boldsymbol{Q}$, as in Chapter 2, represents the vector of adjusted line totals, given by $\boldsymbol{Q} = \boldsymbol{T} - k^{-1}N_d\boldsymbol{B}$, $\boldsymbol{T}$ being the $p \times 1$ vector of line totals and $\boldsymbol{B}$, the vector of block totals; $G_d = (g_{dii'})$ with $g_{dii} = s_{di}$ and for $i \neq i'$, $g_{dii'}$ is the number of times the cross $(i \times i')$ appears in d. Note that under the model (5.5.12), the C-matrix depends on the lines vs blocks incidence matrix as opposed to the customary crosses (treatments) vs blocks incidence matrix.

Gupta and Kageyama (1994) were perhaps the first to consider the problem of finding optimal incomplete block designs for diallel cross experiments, using nested BIB (NBIB) designs. They suggested two families of such designs. More designs for diallel crosses based on NBIB designs were found by Das, Dey and Dean (1998). Following the notation of Section 3.7 (see Definition 3.7.1), consider an NBIB design d_1 with parameters $v = p, b_1, k_1, r, \lambda_1, b_2, k_2 = 2, \lambda_2$ and identify the treatments of the NBIB design with the p lines of a diallel cross experiment. Performing crosses among the lines appearing in the same sub-block (which has size $k_2 = 2$), we get an incomplete block design, say d for a diallel experiment with $p(p-1)/2$ crosses and b_1 blocks, each of size $k_1/2$. Each cross is replicated $2b_2/\{p(p-1)\}$ times in the design. For such a design d,

$$C_d = \frac{2b_1}{p-1}(k-1)(I_p - p^{-1}J_p). \tag{5.5.15}$$

Thus, the design d is variance-balanced for the g.c.a. effects.

Dey and Midha (1996) and Das, Dey and Dean (1998) suggested the use of triangular PBIB designs considered in Chapter 4 for obtaining incomplete block designs for diallel cross experiments. Following the notation of Chapter 4, suppose d_1 is a triangular PBIB design with two associate classes and parameters $v = p(p-1)/2, b, r, k, \lambda_1, \lambda_2, n_1, n_2, p^s_{ij}$, $i, j, s = 1, 2$. Recall that the treatments of d_1 can be indexed by a pair $(i,j), i,j = 1,\ldots,p, i < j$, with a pair of treatments (α, β) and (γ, δ) being ith associates if and only if their (set-theoretic) intersection equals $2 - i$, $i = 1, 2$. From the design d_1, derive a design for diallel crosses by replacing the treatment label (i,j) in d_1 by the cross $(i \times j)$. Suppose $N_d = (n_{dij})$ is the $p \times b$ lines vs blocks incidence matrix of d. Then we have the following result due to Dey and Midha (1996).

Lemma 5.5.1 *For the design d, the following are true:*

(i) $\sum_{j=1}^{b} n_{dij} = r(p-1)$, $\sum_{i=1}^{p} n_{dij} = 2k$,

(ii) $\sum_{j=1}^{b} n_{dij}^2 = r(p-1) + (p-1)(p-2)\lambda_1,\ 1 \le i \le p,$

(iii) $\sum_{j=1}^{b} n_{dij} n_{di'j} = r + (p-2)\{3\lambda_1 + (p-3)\lambda_2\}, 1 \le i, i' \le p, i \ne i'.$

Using Lemma 5.5.1, it is easy to see that for the design d,

$$C_d = \theta(I_p - p^{-1}J_p), \tag{5.5.16}$$

where

$$\theta = \frac{p}{k}\{r(k-1) - (p-2)\lambda_1\}. \tag{5.5.17}$$

From (5.5.16), it is easily seen that the design d for a diallel cross experiment based on a triangular design is again variance-balanced for the g.c.a. effects. This is not surprising as a triangular PBIB design with parameters $v = p(p-1)/2, b, r, k, \lambda_1, \lambda_2$ and treatments indexed by a pair $(i,j), 1 \le i < j \le p$, can be viewed as a nested incomplete block design with p treatments, b blocks of size $2k$ and sub-blocks of size two.

We conclude this section by giving an example of a design for diallel crosses derived from a triangular PBIB design.

Example 5.5.1 Let $p = 5$ and consider a triangular PBIB design with two associate classes and parameters $v = 10, b = 15, r = 3, k = 2, \lambda_1 = 0, \lambda_2 = 1$. The blocks of this design are as follows, where a treatment is represented by a pair $(ij), 1 \le i < j \le 5$:

$$[(12),(34)]; [(12),(35)]; [(12),(45)]; [(13),(24)]; [(13),(25)];$$

$$[(13),(45)]; [(14),(23)]; [(14),(25)]; [(14),(35)]; [(15),(23)];$$

$$[(15),(24)]; [(15),(34)]; [(23),(45)]; [(24),(35)]; [(25),(34)].$$

Using the above design, a design for diallel cross experiment can be obtained simply by replacing each treatment of the type (ij) by the cross $(i \times j)$. For instance, the first block will have the crosses (1×2) and (3×4).

Remark 5.5.1 When the number of lines p is large, the number of crosses in a complete diallel cross experiment may become prohibitively large and in such a case, one might use a "sample" of the crosses only, leading to what are known as *partial diallel crosses.* In the context of partial diallel cross experiments, there are two problems that need to be tackled, viz., (i) finding a "good" sample in the unblocked situation and (ii) arrange the sampled crosses in an incomplete block design which yields efficient estimates of contrasts belonging to (say) the g.c.a. effects. These problems have not yet been solved in their entire generality. We refer to Mukerjee (1997) and Das, Dean and Gupta (1998) for some advances in this direction.

As stated earlier in this section, most of the results on finding good designs for diallel crosses have been derived under a model that has only the g.c.a. effects but no s.c.a. effects. A model where the s.c.a. effects are ignored cannot always be justified from practical considerations and it might be necessary to consider a model that includes both g.c.a. and s.c.a. effects, even if the primary interest centers around the estimation of g.c.a. effect contrasts alone. We refer to Chai and Mukerjee (1999), Choi, Chatterjee, Das and Gupta (2002) and Das and Dey (2004) for some results in this direction.

5.6 Robust Incomplete Block Designs

Proper designing of an experiment is often a difficult task in practice. Even if an experiment is well planned, eventually certain things can go wrong during the conduct of the experiment or while recording the observations. For instance, one or more observations could be lost accidentally, there may be one (or more) outlier(s), or, there could be a trend present within each block, etc. In that event, it is possible that a highly efficient (or, optimal) design originally chosen for the experiment need not remain so. Designs for which the effect of such aberration(s) is small may be termed *robust.* It is of interest to examine the robustness of designs against missing data, presence of outlier(s) etc. In this section, we examine these issues with reference to incomplete block designs. We also discuss some basic issues relating to incomplete block designs that are orthogonal to polynomial trend effects over units within blocks.

5.6.1 Robustness Against an Outlier

Box and Draper (1975) developed a criterion of robustness of response surface designs against the presence of a single outlier. This criterion was extended by Gopalan and Dey (1976) in the context of block designs.

Consider the usual fixed effects linear model

$$Y = X\beta + \epsilon, \ \ \mathbb{E}(\epsilon) = \mathbf{0}, \ \ \mathbb{D}(\epsilon) = \sigma^2 I, \ \sigma^2 > 0, \tag{5.6.1}$$

where, as in Section A.2, $\boldsymbol{Y}$ is the $n \times 1$ vector of observable random variables corresponding to the observations, X is the $n \times m$ design matrix with $Rank(X) = r(\leq m)$, $\boldsymbol{\beta}$ is the vector of unknown parameters and $\boldsymbol{\epsilon}$ is the vector of random error components. Suppose the uth observation Y_u has added to it an unknown aberration c, making it an outlier. It is not known however, to which observation the unknown aberration is added. Under the model (5.6.1), an unbiased estimator of the parameter σ^2 when no outlier is present is given by (see Section A.2 of the Appendix)

$$\hat{\sigma}^2 = R_0^2/(n-r), \tag{5.6.2}$$

where

$$R_0^2 = \boldsymbol{Y}'\boldsymbol{Y} - \boldsymbol{Y}'X(X'X)^{-}X'\boldsymbol{Y}. \tag{5.6.3}$$

When the uth observation has added to it the aberration c, the observations vector gets transformed to $\boldsymbol{Y}_c = \boldsymbol{Y} + c\boldsymbol{e}_u$, where $\boldsymbol{e}_u$ is an $n \times 1$ vector with its uth entry equal to 1 and all other entries equal to zero. Analogous to (5.6.2), if we now estimate σ^2 using

$$\{\boldsymbol{Y}_c'\boldsymbol{Y}_c - \boldsymbol{Y}_c'X(X'X)^{-}X'\boldsymbol{Y}_c\}/(n-r),$$

the bias (or "discrepancy") in estimating σ^2 can be seen to be

$$d_u = c^2 a_{uu}/(n-r), \tag{5.6.4}$$

where a_{uu} is the (u,u)th entry of the matrix $A = \mathrm{pr}^{\perp}(X) = I - X(X'X)^{-}X'$.

Assume now that it is equally likely that the aberration c could occur with any of the n observations. Then the discrepancies are $d_1, \ldots, d_n$ and their average is $\bar{d} = c^2/n$. Thus, the average discrepancy is fixed for a fixed number of observations. In order that no unduly large discrepancy in the estimator of σ^2 is caused by the outlier, it is desirable that the d_u's be as uniform as possible. One measure of such uniformity is reflected in the variance of the d_u's, given by

$$\mathrm{Var}(d) = \frac{c^4(p - r^2/n)}{n(n-r)^2}, \tag{5.6.5}$$

where $p = \sum_{u=1}^{n} p_{uu}^2$ and p_{uu} is the uth diagonal element of the matrix $P = \mathrm{pr}(X) = X(X'X)^{-}X'$. For fixed r, n, minimization of $\mathrm{Var}(d)$ implies minimization of p which in turn means that all p_{uu}'s must be equal. A design may be called *robust* against the presence of a single outlier if under that design, $p_{11} = p_{22} = \cdots = p_{nn}$.

Using this criterion, Gopalan and Dey (1976) showed that the following block designs are robust:
(i) All randomized complete block designs;
(ii) all BIB designs;
(iii) all connected non-group divisible two associate PBIB designs satisfying $\lambda_2 = 0$;
(iv) all semi-regular group divisible designs;
(v) all two-associate triangular designs with $v = m(m-1)/2$ treatments satisfying $r + (m-4)\lambda_1 - (m-3)\lambda_2 = 0$;
(vi) all L_2 PBIB designs with $v = t^2$ treatments satisfying $r+(t-2)\lambda_1 - (t-1)\lambda_2 = 0$.

5.6.2 Robustness Against Missing Data

The robustness of incomplete block designs against non-availability of data has been studied quite extensively; see e.g., Hedayat and John (1974), John (1976), Ghosh (1982), Ghosh, Rao and Singhi (1983), Baksalary and Tabis (1987), Dey and Dhall (1988), Whittinghill (1989), Srivastava, Gupta and Dey (1990), Ghosh, Kageyama and Mukerjee (1992), Dey (1993) and Dey, Midha and Buchthal (1996). Kageyama (1990) presents a review of results in this area till 1988, where some more references can be found. In this subsection, we present some of these results.

We initiate the discussion by describing the notion of *resistant* BIB designs. Suppose d is a BIB design with parameters v, b, r, k, λ and let $\Omega = \{1, 2, \ldots, v\}$ be the set of treatments. Suppose G is a subset of Ω with cardinality $t \leq v-2$, and let d_0 be the design obtained by deleting from d all the experimental units allocated to the treatments in G. Then, we have the following definitions due to Hedayat and John (1974).

Definition 5.6.1 *A BIB design d is said to be globally resistant of degree t if d_0 is variance-balanced with respect to the loss of any subset G of treatments, where the cardinality of G is t.*

Definition 5.6.2 *A BIB design d is said to be locally resistant of degree t if d_0 is variance-balanced with respect to the loss of some (but not all) subsets G of treatments.*

Consider a BIB design d with parameters $v, b, r, k(\geq 3), \lambda$ and let $G = \{i\}$, where $i \in \Omega$. Let d_1^* be the subdesign of d consisting of all blocks that contain i and d_2^* be the subdesign consisting of the blocks of

d that do not contain i. Finally, let d_1 be the design obtained by deleting i from each of the blocks in d_1^*. Hedayat and John (1974) proved the following result.

Theorem 5.6.1 *The design d is locally resistant of degree 1 if and only if d_2^* is a BIB design (or, equivalently, d_1 is a BIB design).*

The above result leads to the following necessary conditions on the design parameters of a locally resistant BIB design of degree one.

Corollary 5.6.1 *If d is a locally resistant BIB design of degree one then its parameters must satisfy the following conditions: (i) $r \geq v - 1$, (ii) $\lambda(k-2)/(v-2)$ is a positive integer, and (iii) $b \geq v + r - 1$.*

For some more results on resistant designs, see Hedayat and John (1974), Most (1975), Baksalary and Puri (1990) and Caliński and Kageyama (2000, Chapter 10).

We next present some more results on the robustness of designs against missing data. A criterion of robustness was considered by Ghosh (1982) with reference to a BIB design. This criterion can be extended to other incomplete block designs in an obvious manner.

Definition 5.6.3 *A balanced incomplete block (BIB) design is said to be robust against the nonavailability of $t(> 0)$ observations if the block design obtained by omitting any t observations from the BIB design remains connected.*

Based on the above criterion, which we shall henceforth call Criterion I, Ghosh (1982) proved the following results.

Theorem 5.6.2 *A BIB design with parameters v, b, r, k, λ is robust against the nonavailability of any $t \leq r - 1$ observations.*

Theorem 5.6.3 *A BIB design is robust against the nonavailability of all observations in $b \leq r - 1$ blocks.*

Similar results in the context of PBIB designs were obtained by Ghosh *et al.* (1983). See also Srivastava, Gupta and Dey (1990) for more results on the robustness of designs based on Criterion I. More general results on the robustness of incomplete block designs based on Criterion I were obtained by Dey (1993) and we describe some of these now.

Consider a connected, binary, proper block design d_0 involving v treatments, b blocks and constant block size k. Let $t(\geq 1)$ observations be missing and let all these t observations pertain to the same treatment. Without loss of generality, one can assume that the missing observations pertain to the first treatment in the first t blocks. We also assume that the t "affected blocks" are not all identical. Let d_t be the residual design obtained by deleting the t observations from d_0 and let N_{d_0} (respectively, N_{d_t}) be the incidence matrix of d_0 (respectively, d_t). Then, one can write N_{d_0} and N_{d_t} as

$$N_{d_0} = \begin{bmatrix} \mathbf{1}_t' & \boldsymbol{e}' \\ F & M \end{bmatrix}, \quad N_{d_t} = \begin{bmatrix} \mathbf{0}' & \boldsymbol{e}' \\ F & M \end{bmatrix},$$

where $\boldsymbol{e}$, F and M are (0,1) matrices of orders $(b-t) \times 1$, $(v-1) \times t$ and $(v-1) \times (b-t)$, respectively. If C_0 and C_t are the C-matrices of d_0 and d_t, respectively, then it can be seen that

$$C_0 = C_t + UU', \tag{5.6.6}$$

where the $v \times t$ matrix U is given by

$$U = \{k(k-1)\}^{-1/2} \begin{bmatrix} (k-1)\mathbf{1}_t' \\ -F \end{bmatrix}. \tag{5.6.7}$$

Based on the above, Dey (1993) proved the following results.

Theorem 5.6.4 *The design d_0 is robust against the loss of any $t(\geq 1)$ observations pertaining to the same treatment according to Criterion I if and only if the matrix $I_t - U'C_0^- U$ is positive definite.*

Corollary 5.6.2 *The design d_0 is robust against the loss of any single observation according to Criterion I if and only if $\boldsymbol{u}'C_0^-\boldsymbol{u} < 1$ where $\boldsymbol{u}' = \{k(k-1)\}^{-1/2}(k-1, -\boldsymbol{f}')$ and $\boldsymbol{f}$ is the vector representing the incidence of the $v-1$ "unaffected" treatments in the block containing the missing observation.*

The result of Corollary 5.6.2 was also obtained by Ghosh *et al.* (1992) using a different approach than that of Dey (1993).

Theorem 5.6.5 *The design d_0 is robust according to Criterion I against the loss of $t(> 1)$ observations pertaining to the same treatment if t does not exceed the smallest positive eigenvalue of C_0.*

Theorem 5.6.6 *The design d_0 is robust according to Criterion I against the loss of all observations in a block if and only if $I_k - V'C_0^- V$ is positive definite where the $v \times k$ matrix V is given by $V' = (I_k - k^{-1}J_k, \mathbf{0})$.*

Theorem 5.6.7 *The design d_0 is robust according to Criterion I against the loss of all observations in a block if the smallest positive eigenvalue of C_0 is larger than unity.*

Using the above results, Dey (1993) showed that the following incomplete block designs are robust (as per Criterion I) against the loss of all observations in a block:

(i) All BIB designs;

(ii) all group divisible designs with the exception of the design with parameters $v = 4 = b, r = 2 = k, m = 2 = n, \lambda_1 = 0, \lambda_2 = 1$;

(iii) all triangular designs with the exception of the design with parameters $v = 10, b = 15, r = 3, k = 2, \lambda_1 = 0, \lambda_2 = 1$;

(iv) all L_i-type PBIB designs, $(i \geq 2)$, with the exception of L_2 design with parameters $v = t^2, b = 2t, r = 2, k = t, \lambda_1 = 1, \lambda_2 = 0$;

(v) all PBIB designs based on partial geometries with more than two replications.

In contrast to Criterion I considered so far, another criterion (which may be called Criterion II) of robustness has also received attention in the literature. This criterion is based on the assessment of loss in efficiency of the residual design when some observations are missing. A design for which this loss is small is termed robust as per Criterion II. The papers by John (1976), Dey and Dhall (1988), Whittinghill (1989), Mukerjee and Kageyama (1990), Ghosh *et al.* (1992) and Dey *et al.* (1996) are all in this spirit. In particular, Dey (1993) showed that the efficiency (E) of the residual design when a single observation is lost from an arbitrary connected, proper incomplete block design is given by

$$E \geq 1 - \frac{H}{H + (v-1)\theta_1(\theta_1 - 1)}, \tag{5.6.8}$$

where θ_1 is the smallest positive eigenvalue and H is the harmonic mean of the positive eigenvalues of the C-matrix of the original design. It was also observed by Dey (1993) that the above lower bound to the efficiency is quite high for most of the two-associate PBIB designs. For similar results on efficiency of the residual design when an arbitrary number of observations are lost from a block, we refer to Dey *et al.* (1996). For more results on the robustness of incomplete block designs against missing data, see Das and Kageyama (1992), Gupta and Srivastava (1992) and Xiaoping and Kageyama (1995).

5.6.3 Trend-free Designs

In certain experimental situations, the response is dependent on the spatial or temporal position of the experimental unit within a block. In such situations, it might be reasonable to assume a common polynomial trend of a specified degree over units within the blocks. When trend effects are present, it is of interest to look for block designs which are "orthogonal" to trend effects and, if such designs are identified, then one can carry out the analysis of the data in the usual manner, as if no trend effects are present. Such designs are called *trend-free*. In this subsection, we study briefly some important aspects of trend-free block designs.

A systematic study of trend-free block designs was initiated by Bradley and Yeh (1980). Consider a binary, proper block design d with v treatments and b blocks, each of size k. Suppose the within-block trend effects are represented by orthogonal polynomials of degree $p < k$. The model postulated is then given by

$$\boldsymbol{Y} = \mu \mathbf{1}_n + D'_{1d}\boldsymbol{\tau} + D'_{2d}\boldsymbol{\beta} + Z\boldsymbol{\theta} + \boldsymbol{\epsilon}. \tag{5.6.9}$$

This is the same fixed effects model (2.2.2) considered in Chapter 2, except for the additional term $Z\boldsymbol{\theta}$, which represents the trend effects. Let the observations in $\boldsymbol{Y}$ collected through the design d be arranged block-wise, i.e., the first k observations in $\boldsymbol{Y}$ come from the first block, the next k come from the second block, and so on. Under this ordering, as observed in Chapter 2, we have

$$D'_{2d} = I_b \otimes \mathbf{1}_k. \tag{5.6.10}$$

The $n \times p$ matrix Z, where $n = bk$, is given by

$$Z = \mathbf{1}_b \otimes F, \tag{5.6.11}$$

where F is a $k \times p$ matrix with columns representing the normalized orthogonal polynomials. It follows then that

$$\mathbf{1}'F = \mathbf{0} \text{ and } F'F = I_p, \tag{5.6.12}$$

and hence,

$$Z'Z = bI_p. \tag{5.6.13}$$

According to Bradley and Yeh (1980), a block design is called trend-free if the sum of squares due to treatments eliminating block effects, under the model (5.6.9) is the same as obtained under the model (2.2.2). We now derive a necessary and sufficient condition for a block design to be trend-free. To that end, let

$$\begin{aligned} X_{1d} &= [\mathbf{1}_n \; D'_{2d} \; Z], \\ X_{2d} &= [\mathbf{1}_n \; D_{1d}], \\ A_{id} &= I_n - X_{id}(X'_{id}X_{id})^- X'_{id}, \; i = 1, 2, \\ G_{id} &= D_{1d}A_{id}D'_{id}, \; i = 1, 2. \end{aligned} \tag{5.6.14}$$

It can then be seen easily that

$$X'_{1d}X_{1d} = \begin{bmatrix} n & k\mathbf{1}' & \mathbf{0}' \\ k\mathbf{1} & kI_b & \mathbf{0}' \\ \mathbf{0} & \mathbf{0} & bI_p \end{bmatrix}, \quad X'_{2d}X_{2d} = \begin{bmatrix} n & k\mathbf{1}' \\ k\mathbf{1} & kI_b \end{bmatrix}. \tag{5.6.15}$$

A g-inverse of $X'_{1d}X_{1d}$ and $X'_{2d}X_{2d}$ is given, respectively, by

$$(X'_{1d}X_{1d})^- = \begin{bmatrix} 0 & \mathbf{0}' & \mathbf{0}' \\ \mathbf{0} & k^{-1}I_b & \mathbf{0}' \\ \mathbf{0} & \mathbf{0} & b^{-1}I_p \end{bmatrix}, \quad (X'_{2d}X_{2d})^- = \begin{bmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & k^{-1}I_b \end{bmatrix}. \tag{5.6.16}$$

Therefore,

$$A_{1d} = I_n - k^{-1}D'_{2d}D_{2d} - b^{-1}ZZ', \; A_{2d} = I_n - k^{-1}D'_{2d}D_{2d}. \tag{5.6.17}$$

It can be shown that

$$\begin{aligned} T_t &= \text{Adjusted treatment S.S. under model (5.6.9)} \\ &= \mathbf{Y}'A_{1d}D'_{1d}G^-_{1d}D_{1d}A_{1d}\mathbf{Y} \end{aligned} \tag{5.6.18}$$

and

$$\begin{aligned} T_0 &= \text{Adjusted treatment S.S. under model (2.2.2)} \\ &= \mathbf{Y}'A_{2d}D'_{1d}G^-_{2d}D_{1d}A_{2d}\mathbf{Y}. \end{aligned} \tag{5.6.19}$$

Also, the unadjusted block sum of squares under (5.6.9) remains the same as under (2.2.2) as expected, since the trend effects add up to zero over units within each block. Now, a design is trend-free if and only if $T_t = T_0$. First assume that $T_t = T_0$. Then, this implies that

$$\begin{aligned} & A_{d1}D'_{1d}G^-_{1d}D_{1d}A_{1d} = A_{2d}D'_{1d}G^-_{2d}D_{1d}A_{2d} \\ \Rightarrow\; & D_{1d}A_{d1}D'_{1d}G^-_{1d}D_{1d}A_{1d}D'_{1d} = D_{1d}A_{2d}D'_{1d}G^-_{2d}D_{1d}A_{2d}D'_{1d} \\ \Rightarrow\; & D_{1d}A_{1d}D'_{1d} = D_{1d}A_{2d}D'_{1d} \\ \Rightarrow\; & D_{1d}(A_{2d} - A_{1d})D'_{1d} = \mathbf{0}. \end{aligned} \tag{5.6.20}$$

Using (5.6.17) and recalling that $D_{1d}D'_{2d} = N_d$, the incidence matrix of d, it can be seen that $T_t = T_0$ if

$$D_{1d}Z = \mathbf{0}. \tag{5.6.21}$$

Conversely, if (5.6.21) holds, then one can show, using (5.6.17) that $T_t = T_0$. We thus have the following result due to Bradley and Yeh (1980).

Theorem 5.6.8 *A necessary and sufficient condition for a block design to be trend-free is that (5.6.21) holds.*

Here is an example of a trend-free balanced incomplete block design with parameters $v = 6, b = 10, r = 5, k = 3, \lambda = 2$ and $p = 1$. Here, the columns represent the blocks.

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 & 0 & 5 & 5 & 0 & 5 \\ 2 & 3 & 4 & 0 & 1 & 2 & 3 & 4 & 5 & 1 \end{matrix}.$$

For several other interesting results on trend-free block designs, see Yeh and Bradley (1983), Stufken (1988), Chai and Majumdar (1993), Chai (1995), Jacroux, Majumdar and Shah (1995, 1997) and Chai and Stufken (1999).

5.7 Exercises

5.1. Provide a proof of Lemma 5.2.1.

5.2. Give proofs for the expressions given in (5.2.15) and (5.2.16).

5.3. Consider the following design for a 2^3 factorial:

Block I: (000,010,100,110); Block II: (001,011,101,111).

Which of the factorial effect(s) are estimable? Does the design have OFS?

5.4. Consider the following design for a 3×4 factorial having 12 blocks (denoted by $B_1, \ldots, B_{12}$) of size three each:

B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}
00	00	00	01	01	01	02	02	02	03	03	03
11	12	13	10	12	13	10	11	13	10	11	12
22	23	21	23	20	22	21	23	20	22	20	21

Show that this design has both OFS and balance.

5.5. For a 3×2^2 factorial experiment, the following design was used, using 6 blocks of size 6 each:

Block I	Block II	Block III	Block IV	Block V	Block VI
000	010	000	010	000	010
100	110	100	101	100	110
200	210	200	201	200	210
001	011	010	011	011	001
101	111	110	111	111	101
201	211	210	211	211	202

Examine whether or not the above design is a balanced factorial design.

5.6. Show that for a symmetric parallel line assay with m doses, an L-design with block size k exists if and only if $\frac{1}{2}k(m+1)$ is even.

5.7. Give a proof of Lemma 5.5.1.

5.8. Show that an orthogonal block design d with incidence matrix $N_d = (n_{dij})$ cannot be robust against the presence of a single outlier unless all the n_{dij}'s are equal.

5.9. Using Corollary 5.6.1, show that an affine resolvable BIB design cannot be locally resistant of degree one, except possibly the designs belonging to the family with parameters $v = 4t, b = 2(4t-1), r = 4t-1, k = 2t, \lambda = 2t-1$, where $t \geq 1$ is an integer.

5.10. Verify the statement in (5.6.6).

5.11. Verify the expressions for the adjusted treatment sum of squares as given in (5.6.18) and (5.6.19).

5.12. Examine whether the following design involving $v = 5$ treatments and $b = 10$ blocks is linear trend-free (i.e., for $p = 1$):

$$(1,2,3);(1,2,4);(1,2,5);(3,4,1);(3,5,1);$$
$$(4,5,1);(4,2,3);(5,2,3);(5,2,4);(3,4,5).$$

Chapter 6

Optimality Aspects of Block Designs

6.1 Introduction

In this chapter, we review some results on the optimality of incomplete block designs. The coverage here is not complete; rather, the material presented in this chapter may be viewed as an invitation to the vast area of vigorous research. It is hoped that the information provided here will be useful to research workers who are new to this area. For a more comprehensive account of the developments in optimal block designs, we refer to Shah and Sinha (1989) and for an authoritative and more recent review of optimal designs (including block designs) based on exact theory, a reference may be made to Cheng (1996).

A systematic study of optimal designs in a very broad context was initiated by Kiefer (1958) and we refer the reader to Shah and Sinha (1989) and Pukelsheim (1993) for excellent accounts of various optimality criteria and optimal designs. Pukelsheim (1993) deals mostly with what is known as *approximate theory*, in which each probability measure on the experimental region is considered as a design, where the probabilities represent the proportion of observations at different sites, and the general problem is to determine these proportions "optimally". In contrast, the *exact theory* is concerned with the problem of determining an optimal design for a given finite number of observations. It turns out that the approximate theory is more appropriate for regression design problems while the exact theory seems to be more relevant for design problems in discrete settings, like that of a block design.

In Section 6.2, we describe some important optimality criteria. Results on optimal proper block designs for inference on a complete set of orthonormal treatment contrasts are reviewed in Section 6.3. Optimal designs for inference on control-test comparisons are described in

Section 6.4. In Section 6.5, some aspects of optimal incomplete block designs for parallel line assays are covered. Finally, in Section 6.6, optimal incomplete block designs for diallel crosses are considered.

6.2 Optimality Criteria

Recall that a block design involving v treatments and b blocks, each of size k say, is an allocation of v treatments to the $n = bk$ experimental units. For given values of the design parameters v, b, k, typically there will be several choices for the design and these alternative designs form a class of competing designs. To discriminate among different designs belonging to a class of competing designs, one needs to compare the designs under a suitable model postulated for the observations generated by the designs and some well-defined criterion, which depends on the objective of the study. For instance, in the context of incomplete block designs, often the objective is to compare the treatment effects and then, one would choose a design that provides, in some meaningful sense, best estimates of treatment contrasts under the postulated model. If such a design is identified, then it is called *optimal* under the given criterion and the model.

Consider a block design d involving v treatments, b blocks each of (constant) size k and as in Section 2.2 of Chapter 2, assume the fixed effects additive linear model (2.2.1) for the observations generated by d. Recall from Chapter 2 that the reduced (intra-block) normal equations for estimating linear functions of treatment effects under the design d are given by

$$C_d\boldsymbol{\tau} = \boldsymbol{Q}. \tag{6.2.1}$$

Suppose the inference problem is specified as

$$\mathcal{P} : \boldsymbol{\theta} = L\boldsymbol{\tau},$$

where L is a $p \times v$ matrix with $L\mathbf{1}_v = \mathbf{0}$ and $Rank(L) = p$. Clearly, $\boldsymbol{\theta}$ represents a set of p linearly independent treatment contrasts. Suppose that d allows the estimability of each of the components of $\boldsymbol{\theta}$. Let $\hat{\boldsymbol{\theta}}_d$ represent the BLUEs of the components in $\boldsymbol{\theta}$ under d and $V_d = \mathbb{D}(\hat{\boldsymbol{\theta}}_d)$, the dispersion matrix of $\hat{\boldsymbol{\theta}}_d$. Then it is reasonable to choose a design belonging to a class of competing designs for which the corresponding dispersion matrix is "small" in some meaningful sense.

Let $\mathcal{D}$ be the class of all block designs with v treatments and b blocks each of size k such that each member of $\mathcal{D}$ keeps $\boldsymbol{\theta}$ estimable. We then have the following definitions.

Definition 6.2.1 *A design $d^* \in \mathcal{D}$ is said to be A-optimal over $\mathcal{D}$ if*

$$\mathrm{tr}(V_{d^*}) \leq \mathrm{tr}(V_d), \text{ for any other design } d \in \mathcal{D}.$$

Clearly, if $d^* \in \mathcal{D}$ is A-optimal over $\mathcal{D}$ then d^* minimizes the average variance of the BLUEs of the components of $\boldsymbol{\theta}$ over the class of competing designs $\mathcal{D}$.

Definition 6.2.2 *A design $d^* \in \mathcal{D}$ is said to be D-optimal over $\mathcal{D}$ if*

$$\det(V_{d^*}) \leq \det(V_d), \text{ for any other design } d \in \mathcal{D}.$$

The D-optimality criterion has the following statistical significance. Under the assumption of normality of the errors, $\hat{\boldsymbol{\theta}}_d$ has a normal distribution. Suppose we are interested in the $100(1-\alpha)\%$ confidence ellipsoid for $\boldsymbol{\theta}$. In such a case, one would like to choose a design from the competing class of designs for which the volume of the confidence ellipsoid is as small as possible. Since the volume of the confidence ellipsoid is proportional to the square root of the determinant of V_d, a D-optimal design minimizes the volume of the confidence ellipsoid over the class of competing designs.

Consider now the problem of inference on a complete set of *orthonormal* treatment contrasts. Clearly, in this case, we must restrict attention to the class of connected designs, i.e., now $\mathcal{D}$ consists of all connected block designs involving v treatments and b blocks each of size k. The inference problem can now be specified as

$$\mathcal{P} : \boldsymbol{\theta} = P\boldsymbol{\tau},$$

where P is a $(v-1) \times v$ matrix such that the matrix

$$A = \begin{pmatrix} \frac{1}{\sqrt{v}}\mathbf{1}'_v \\ P \end{pmatrix} \tag{6.2.2}$$

is orthogonal. It follows then that

$$P\mathbf{1}_v = \mathbf{0}, \;\; PP' = I_{v-1}, \;\; \text{and} \;\; P'P = I_v - v^{-1}J_v. \tag{6.2.3}$$

If $P\hat{\tau}$, where $\hat{\tau}$ is a solution of (6.2.1), is the BLUE of $P\tau$, then the dispersion matrix of $P\hat{\tau}$ is given by

$$\mathbb{D}(P\hat{\tau}) = \sigma^2 PC_d^- P' = \sigma^2 PC_d^+ P', \tag{6.2.4}$$

where C_d^-(respectively, C_d^+) is an arbitrary (respectively, the Moore-Penrose) generalized inverse of C_d. Note that PC_d^-P' is invariant with respect to the choice of a generalized inverse. We now have the following result.

Lemma 6.2.1 *For a connected design d,*

$$PC_d^- P' = PC_d^+ P' = (PC_dP')^{-1}.$$

Also, it is not hard to see that the eigenvalues of PC_d^+P' and the positive eigenvalues of C_d^+ are the same and, the positive eigenvalues of C_d^+ are the reciprocals of the positive eigenvalues of C_d. Therefore, it is easier to work with C_d directly, instead of $PC_dP' = (PC_d^+P')^{-1}$, which by virtue of Lemma 6.2.1 is the information matrix of $P\tau$. Though rigorously speaking, the information matrix of $P\tau$ under a connected design d is $(PC_d^-P')^{-1}$, in the following we continue to call C_d as the information matrix for treatment effects under the design d. If $\lambda_{d1} \leq \ldots \leq \lambda_{d,v-1}$ are the positive eigenvalues of C_d, then it follows that the A- and D-criteria in the context of inference on a complete set of orthonormal treatment contrasts take the following forms:

A-optimality: Minimize $\sum_{i=1}^{v-1} \lambda_{di}^{-1}$.

D-optimality: Minimize $\prod_{i=1}^{v-1} \lambda_{di}^{-1}$ or, equivalently, maximize $\prod_{i=1}^{v-1} \lambda_{di}$.

Remark 6.2.1 In Lemma 2.5.1, it was shown that the average variance of the BLUEs of all elementary treatment contrasts is inversely proportional to the harmonic mean of the positive eigenvalues of the C-matrix. Therefore, an A-optimal design also minimizes this average variance.

Another commonly used optimality criterion, the E-criterion, calls for the minimization of the largest eigenvalue of C_d^+ (or, equivalently, the maximization of the smallest positive eigenvalue of C_d). Let $\boldsymbol{p}'\boldsymbol{\tau}$ be a treatment contrast. The variance of the BLUE of $\boldsymbol{p}'\boldsymbol{\tau}$ under a design d is

$$\text{Var}(\boldsymbol{p}'\hat{\boldsymbol{\tau}}) = \sigma^2 \boldsymbol{p}'C_d^+\boldsymbol{p}.$$

Therefore,

$$\max_{p:p'p=1} \frac{\mathrm{Var}(p'\hat{\tau})}{\sigma^2} = \lambda_{max}(C_d^+),$$

where $\lambda_{max}(C_d^+)$ is the largest eigenvalue of C_d^+. Since the positive eigenvalues of C_d^+ are the reciprocals of the positive eigenvalues of C_d, it follows that

$$\max_{p:p'p=1} \frac{\mathrm{Var}(p'\hat{\tau})}{\sigma^2} = \lambda_{d1}^{-1}.$$

This leads to the E-criterion in the sense that an E-optimal design minimizes the maximum variance of a normalized treatment contrast. The E-criterion has another statistical interpretation. Suppose the observation vector follows a multivariate normal distribution and we wish to test the hypothesis $H : \tau_1 = \cdots = \tau_v$, where for $1 \le i \le v$, τ_i denotes the effect of the ith treatment. The usual analysis of variance F-test for this testing problem has a power function depending monotonically (increasing) on a parameter $\xi = \sigma^{-1}\tau' C_d \tau$ and thus, under the assumption of normality, an E-optimal design maximizes the minimum power of the associated F-test of size α on the contour $\tau'\tau = c$ for every α and c.

Kiefer (1975) introduced a class of optimality criteria, defined as

$$\phi_p(C_d) = \left\{ \sum_{i=1}^{v-1} \lambda_{di}^{-p}/(v-1) \right\}^{1/p}, \quad 0 < p < \infty. \tag{6.2.5}$$

A design $d^* \in \mathcal{D}$ is called ϕ_p-optimal over a class of competing designs $\mathcal{D}$ if

$$\phi_p(C_{d^*}) \le \phi_p(C_d) \text{ for any other } d \in \mathcal{D}.$$

Clearly, the A-criterion is a ϕ_p-criterion with $p = 1$. It can be shown that the D-criterion is a point-wise limit of ϕ_p-criteria as $p \to 0$ and the point-wise limit of ϕ_p as $p \to \infty$ gives the E-criterion.

Cheng (1978) considered a class of optimality criteria, called the type I criteria, which involves the minimization of $\psi_f(C_d) = \sum_{i=1}^{v-1} f(\lambda_{di})$, where f is a convex, non-increasing function defined over $(0, M_0)$ where $M_0 = \max \mathrm{tr}(C_d)$, the maximum being taken over the class of relevant designs under consideration. The function f is assumed to satisfy the following conditions:

(i) f is continuously differentiable over $(0, M_0)$;

(ii) f' is strictly concave, i.e., $f' < 0$, $f'' > 0$, $f''' < 0$ over $(0, M_0)$;

(iii) f is continuous at 0 and $f(0) = \lim_{x \to 0^+} f(x) = \infty$.

Condition (ii) above of concavity of the derivative function f' is imposed because without this condition, the class of optimality criteria becomes too large and thus, it is hard to find a design that is optimal in such a strong sense. Condition (iii) ensures that designs for which the eigenvalues of the C-matrix are close to zero cannot be optimal. Cheng (1978) also defined generalized type I criteria as a point-wise limit of a sequence of type I criteria.

It may be noted that the A- and D-optimality criteria are included in the type I class of criteria; choosing $f(x) = x^{-1}$ gives the A-criterion and $f(x) = -\log(x)$ gives the D-criterion. The ϕ_p criteria are also included in the type I criteria and the E-criterion is included in the generalized type I criteria.

The notion of *universal optimality* introduced by Kiefer (1975) helps in unifying the various optimality criteria. Let $\mathcal{B}_{v,0}$ be the set of all $v \times v$ symmetric matrices with zero row (column) sums. Consider the class Φ of real-valued functions $\phi(\cdot)$ defined on $\mathcal{B}_{v,0}$, such that
(a) $\phi(\cdot)$ is convex,
(b) $\phi(bC)$ is a nonincreasing function of $b \geq 0$ for any $C \in \mathcal{B}_{v,0}$, and
(c) $\phi(\cdot)$ is invariant under each simultaneous permutation of rows and columns.

A design d^* in a class of competing designs $\mathcal{D}$ is said to be universally optimal over $\mathcal{D}$ if for *each* $\phi(\cdot) \in \Phi$, $\phi(C_{d^*}) \leq \phi(C_d)$, for any other design $d \in \mathcal{D}$. It can be shown that a design that is universally optimal is also A-, D- and E-optimal.

The following result due to Kiefer (1975) provides a sufficient condition for determining a universally optimal design.

Theorem 6.2.1 *Suppose a class $\mathcal{C} = \{C_d : d \in \mathcal{D}\}$ of matrices in $\mathcal{B}_{v,0}$ contains a C_{d^*} for which*
(i) C_{d^*} is completely symmetric, and
(ii) $\operatorname{tr}(C_{d^*}) = \max_{d \in \mathcal{D}} \operatorname{tr}(C_d)$.
Then d^* is universally optimal over $\mathcal{D}$.

Proof. Let d^* be *not* universally optimal in $\mathcal{D}$. Then there exists a design $d_1 \in \mathcal{D}$ such that

$$\phi(C_{d_1}) < \phi(C_{d^*}), \text{ for some } \phi \in \Phi. \tag{6.2.6}$$

Let πC_{d_1} be obtained from C_{d_1} by simultaneously permuting the rows and columns of C_{d_1} according to the permutation π and let

$$\bar{C}_{d_1} = \frac{1}{v!}\sum_{\pi} \pi C_{d_1},$$

where the sum is over all the $v!$ possible permutations. Then, by condition (a) of the universal optimality criterion $\phi(\cdot)$, we have

$$\begin{aligned}
\phi(\bar{C}_{d_1}) &= \phi\left(\frac{1}{v!}\sum_{\pi} \pi C_{d_1}\right) \\
&\leq \sum_{\pi} \frac{1}{v!}\phi(\pi C_{d_1}) \\
&= \sum_{\pi} \frac{1}{v!}\phi(C_{d_1}) \\
&= \phi(C_{d_1}), \text{ using condition (c) on } \phi(\cdot). \qquad (6.2.7)
\end{aligned}$$

From (6.2.6) and (6.2.7), we thus have

$$\phi(C_{d^*}) > \phi(C_{d_1}) \geq \phi(\bar{C}_{d_1}). \qquad (6.2.8)$$

Also, it is not hard to see that $\bar{C}_{d_1}$ is a completely symmetric matrix and is in $\mathcal{B}_{v,0}$ and hence is of the form αC_{d^*} for some $\alpha \geq 0$. Furthermore, since $\text{tr}(\bar{C}_{d_1}) = \text{tr}(C_{d_1})$, we have from (ii) of the theorem,

$$\text{tr}(C_{d^*}) \geq \text{tr}(C_{d_1}) = \text{tr}(\bar{C}_{d_1}) = \text{tr}(\alpha C_{d^*}),$$

which shows that $\alpha \leq 1$. But by condition (b) on $\phi(\cdot)$ and (6.2.8),

$$\phi(C_{d_1}) \geq \phi(\bar{C}_{d_1}) = \phi(\alpha C_{d^*}) \geq \phi(C_{d^*}), \text{ as } \alpha \leq 1.$$

This contradicts (6.2.6) and the proof is complete. □

The above result of Kiefer, though useful in many cases, has some limitations. When the class of information matrices $\mathcal{C}$ does not contain any completely symmetric C-matrix with maximum trace, Theorem 6.2.1 is not useful. Yeh (1986) generalized Theorem 6.2.1 and proved the following result.

Theorem 6.2.2 *Suppose a class $\mathcal{C} = \{C_d : d \in \mathcal{D}\}$ of matrices in $\mathcal{B}_{v,0}$ contains a C_{d^*} such that*
(i) for any $d \in \mathcal{D}, C_d \neq \mathbf{0}$, there exist scalars $a_{di} \geq 0$, $1 \leq i \leq m$ satisfying

$$C_{d^*} = \sum_{i=1}^{m} a_{di} P_i C_d P_i', \qquad (6.2.9)$$

(ii) $\mathrm{tr}(C_{d^*}) = \max\limits_{d \in \mathcal{D}} \mathrm{tr}(C_d)$,
where $m = v!$ *and* $P_1, P_2, \ldots, P_m$ *are the permutation matrices of order* v. *Then,* d^* *is universally optimal in* $\mathcal{D}$.

Note that Theorem 6.2.1 is a special case of Theorem 6.2.2, as we can write C_{d^*} in the form (6.2.9) by taking $a_{di} = \mathrm{tr}(C_{d^*})/\{m\mathrm{tr}(C_d)\}$ for each i. Theorem 6.2.2 is found useful, for example, in determining universally optimal binary incomplete block designs when $b = vu \pm 1$ and $k = v - 1$, where u is a positive integer.

Remark 6.2.2 An extension of universal optimality criterion of Kiefer (1975) was considered by Shah and Sinha (2006). Let A_d denote the information matrix for the relevant parametric functions under a suitable model using the design d. Let g be a permutation of $\{1, 2, \ldots, v\}$, that is $g \in S_v$, the symmetric group of permutations on $\{1, 2, \ldots, v\}$. As per Shah and Sinha (2006), a design d^* with information matrix C_{d^*} is said to be universally optimal in an appropriate class of competing designs if it minimizes every real-valued optimality functional $\phi(\cdot)$ defined over the set of n.n.d. matrices, that satisfies the following conditions:
(i) $\phi(A_{d_g}) = \phi(A_d)$ for every $g \in S_v$, where d_g is the design obtained by permuting the treatment labels according to g;
(ii) $A_d \geq A_f \Rightarrow \phi(A_d) \leq \phi(A_f)$, where d and f are any two designs in the competing class;
(iii) $\phi(\sum w_g A_{d_g}) \leq \phi(A_d)$, where $\{w_g\}$ are nonnegative rationals satisfying $\sum_g w_g = 1$. Here g runs over all the $v!$ permutations in S_v.

Note that every convex functional satisfies (iii). This formulation of universal optimality is an extension of the original formulation of Kiefer (1975) in the sense that the condition of convexity in the original formulation is replaced by a slightly weaker condition (iii) above.

A sufficient condition for d_0 to be universally optimal (as per the extended definition) is that

$$\sum w_g A_{d_g} \leq A_{d_0} \quad \text{for every } d,$$

where the $\{w_g\}$ can be any specific set of weights (which may depend on d).

A useful optimality criterion, called MV-optimality, was introduced by Takeuchi (1961). The term "MV-optimality" was coined by Jacroux

(1983). Takeuchi (1961) argued that in a block design context, one is primarily interested in elementary treatment contrasts and thus, while seeking an optimal design one might minimize the *maximum* variance of the BLUEs of all elementary treatment contrasts. This criterion is *not* exclusively a function of the positive eigenvalues of the C-matrix, as shown in the following example.

Let $v = 3$ with treatments labeled as 1,2 and 3. Consider two designs d_1 and d_2, whose block contents are given below:

$$d_1 : \ (1,2); (1,2); (1,2); (1,3); (1,3); (1,3).$$

$$d_2 : \ (1,2); (1,2); (1,2); (1,2); (1,3); (2,3).$$

The C-matrices for these two designs are as given below.

$$C_{d_1} = \tfrac{1}{2}\begin{bmatrix} 6 & -3 & -3 \\ -3 & 3 & 0 \\ -3 & 0 & 3 \end{bmatrix}, \quad C_{d_2} = \tfrac{1}{2}\begin{bmatrix} 5 & -4 & -1 \\ -4 & 5 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

The positive eigenvalues of both C_{d_1} and C_{d_2} are $\lambda_1 = 3/2$ and $\lambda_2 = 9/2$. Thus, d_1 and d_2 are equivalent under any optimality criterion which is a sole function of the positive eigenvalues of the C-matrix. However, the maximum variance of the BLUE of elementary treatment contrasts is $1.3334\sigma^2$ for d_1 and $1.1111\sigma^2$ for d_2. Thus, on the basis of the MV-optimality criterion, d_2 is to be preferred over d_1.

It should also be noted that a design which is universally optimal is also MV-optimal.

Another optimality criterion that has received attention in the literature is the (M,S)-optimality criterion. As before, let $\lambda_{d1} \leq \lambda_{d2} \leq \cdots \leq \lambda_{d,v-1}$ be the positive eigenvalues of C_d, the C-matrix of a connected design d. The (M,S)-criterion then involves the following procedure:
(i) Maximize $\mathrm{tr}(C_d) = \sum_{i=1}^{v-1} \lambda_{di}$ over the class of competing designs;
(ii) minimize $\mathrm{tr}(C_d^2) = \sum_{i=1}^{v-1} \lambda_{di}^2$ over the subclass of designs that have maximal $\mathrm{tr}(C_d)$.

A design that satisfies (i) and (ii) above is called (M,S)-optimal. This criterion was proposed originally in the context of block designs by Shah (1960) and Eccleston and Hedayat (1974). As noted by Cheng (1996), the (M,S)-criterion is not an optimally criterion and is rather a procedure for quickly identifying designs that might be optimal or highly efficient with respect to other more meaningful criteria. The rationale behind the (M,S)-optimality criterion is as follows: most of the optimality criteria demand that the positive eigenvalues of C_d, λ_{di}'s, are 'as nearly equal as possible', while their sum should be 'as large as possible'. It is therefore reasonable to choose a design with the least value of $S = \sum_{i=1}^{v-1} \lambda_{di}^2$ from a subclass containing designs with maximum value of $\sum_{i=1}^{v-1} \lambda_{di}$. The main advantage of the (M, S)-criterion is that $\mathrm{tr}(C_d)$ and $\mathrm{tr}(C_d^2)$ are very easy to compute and optimize, these being simple functions of the positive eigenvalues of C_d. It is because of this reason that the (M, S)-optimality criterion is considered as a handy tool in the search for an optimal design.

Bagchi and Bagchi (2001) considered a class of optimality criteria more general than the one considered by Cheng (1978). This is called M-optimality and is based on the concept of majorization. For a comprehensive account of the theory of majorization, see Marshall and Olkin (1979). For a vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ denote the components of $\boldsymbol{x}$ arranged in increasing order, i.e., $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{x}$ is said to be weakly majorized from above by $\boldsymbol{y}$ (written as $\boldsymbol{x} \prec^w \boldsymbol{y}$) if

$$\sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)}, \; k = 1, 2, \ldots, n.$$

A design $d_1 \in \mathcal{D}$ is said to be better than another design $d_2 \in \mathcal{D}$ in the sense of majorization (or, d_1 is M-better than d_2) if

$$\mu(C_{d_1}) \prec^w \mu(C_{d_2}),$$

where for a real symmetric matrix A, $\mu(A)$ is the vector of eigenvalues of A, arranged in increasing order. A design $d^* \in \mathcal{D}$ is called M-optimal over $\mathcal{D}$ if it is M-better than every other member of $\mathcal{D}$. It was shown by Bagchi and Bagchi (2001) that if a design d_1 is M-better than d_2, then d_1 is better than d_2 with respect to every type I criterion and thus, in particular, d_1 is better than d_2 with respect to the A-, D- and E-optimality criteria. It may also be noted that a universally optimal design is M-optimal.

In closing this section, we state the following well known result which will be found useful in the sequel.

Lemma 6.2.2 *For given positive integers s and t, the minimum of $n_1^2 + n_2^2 + \cdots + n_s^2$ subject to $n_1 + n_2 + \cdots + n_s = t$, where n_i's are nonnegative integers, is obtained when $t - s[t/s]$ of the n_i's are equal to $[t/s] + 1$ and $s - t + s[t/s]$ are equal to $[t/s]$, $[z]$ denoting the greatest integer not* *exceeding z. The corresponding minimum of $n_1^2 + \cdots + n_s^2$ is $t(2[t/s] + 1) - s[t/s]([t/s] + 1)$.*

6.3 Optimality of Proper Block Designs

In this section, we describe some important results on the optimality of incomplete block designs that are proper, i.e., have constant block sizes. The discussion is initiated by first considering the optimality of symmetric designs. The optimality of some asymmetric proper incomplete block designs is considered next.

6.3.1 Optimality of Symmetric Designs

A symmetric design is defined to be one whose C-matrix has all its non-zero eigenvalues equal, or equivalently, whose C-matrix is completely symmetric (recall the definition of a completely symmetric matrix from Section A.1 of the Appendix). In the block design set up, a BIB design and a balanced block design are symmetric designs. In this subsection, we present some results on the optimality of BIB and balanced block designs. We shall let $\mathcal{D}(v, b, k)$ to denote the class of all connected block designs with v treatments and b blocks each of size $k \geq 2$. The first result in this direction due to Kiefer (1958) follows (see also Roy (1958) and Mote (1958)).

Theorem 6.3.1 *A BIB design, whenever existent, is A-, D and E-optimal over $\mathcal{D}(v, b, k)$ for inferring on a complete set of orthonormal treatment contrasts.*

Proof. (i) A-optimality.

Let a BIB design d^* exist in $\mathcal{D}(v, b, k)$ and let $d \in \mathcal{D}(v, b, k)$ be arbitrary. For $d \in \mathcal{D}(v, b, k)$, let $\lambda_{d1} \leq \lambda_{d2} \leq \cdots \leq \lambda_{d,v-1}$ be the positive eigenvalues of C_d, $N_d = (n_{duj})$ be the incidence matrix of d and r_{du} be the replication of the uth treatment in d, $1 \leq u \leq v$. By the arithmetic mean-harmonic mean inequality, we have

$$\begin{aligned}(v-1)^{-1}\sum_{i=1}^{v-1}\lambda_{di} &\geq \frac{v-1}{\sum_{i=1}^{v-1}\lambda_{di}^{-1}} \\ \Rightarrow \left(\sum_{i=1}^{v-1}\lambda_{di}^{-1}\right)^{-1} &\leq (v-1)^{-2}\sum_{i=1}^{v-1}\lambda_{di}. \qquad (6.3.1)\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{i=1}^{v-1} \lambda_{di} = \operatorname{tr}(C_d) &= \sum_{u=1}^{v} \left(r_{du} - k^{-1} \sum_{j=1}^{b} n_{duj}^2 \right) \\
&= bk - k^{-1} \sum_{u=1}^{v} \sum_{j=1}^{b} n_{duj}^2 \\
&\leq bk - k^{-1} \sum_{u=1}^{v} \sum_{j=1}^{b} n_{duj} \\
&= bk - k^{-1} bk = b(k-1). \qquad (6.3.2)
\end{aligned}$$

Combining (6.3.1) and (6.3.2), we have a lower bound to $\sum_{i=1}^{v-1} \lambda_{di}^{-1}$ as

$$\sum_{i=1}^{v-1} \lambda_{di}^{-1} \geq \frac{(v-1)^2}{b(k-1)}. \qquad (6.3.3)$$

As noted in Chapter 3, for the BIB design $d^* \in \mathcal{D}(v, b, k)$, $\lambda_{d^*i} = \lambda v/k$, $1 \leq i \leq v-1$, where λ is the usual pairwise concurrence parameter of d^*. Therefore,

$$\begin{aligned}
\sum_{i=1}^{v-1} \lambda_{d^*i}^{-1} &= \frac{k(v-1)}{\lambda v} \\
&= \frac{k(v-1)^2}{vr(k-1)} \\
&= \frac{(v-1)^2}{b(k-1)}
\end{aligned}$$

which equals the lower bound (6.3.3). This proves the A-optimality of d^*.

(ii) D-optimality.

By the arithmetic mean-geometric mean inequality, we have

$$\begin{aligned}
(v-1)^{-1} \sum_{i=1}^{v-1} \lambda_{di} &\geq \left(\prod_{i=1}^{v-1} \lambda_{di} \right)^{\frac{1}{v-1}} \\
\Rightarrow \prod_{i=1}^{v-1} \lambda_{di}^{-1} &\geq \left(\frac{v-1}{\sum_{i=1}^{v-1} \lambda_{di}} \right)^{v-1}
\end{aligned}$$

$$\geq \left(\frac{v-1}{b(k-1)}\right)^{v-1}, \tag{6.3.4}$$

using (6.3.2). For the BIB design $d^* \in \mathcal{D}(v,b,k)$,

$$\begin{aligned}\prod_{i=1}^{v-1} \lambda_{d^*i}^{-1} &= \left(\frac{k}{\lambda v}\right)^{v-1} \\ &= \left(\frac{k(v-1)}{vr(k-1)}\right)^{v-1} \\ &= \left(\frac{v-1}{b(k-1)}\right)^{v-1},\end{aligned}$$

which equals the lower bound (6.3.4). This establishes the D-optimality of d^*.

(iii) E-optimality.

Since $0 < \lambda_{d1} \leq \lambda_{d2} \leq \cdots \leq \lambda_{d,v-1}$, we have

$$\lambda_{d1} \leq \frac{1}{v-1}\sum_{i=1}^{v-1} \lambda_{di}.$$

Using (6.3.2), we have

$$\lambda_{d1} \leq \frac{b(k-1)}{v-1}, \tag{6.3.5}$$

which gives an upper bound for λ_{d1}. For a BIB design $d^* \in \mathcal{D}(v,b,k)$,

$$\lambda_{d^*1} = \lambda v/k = \frac{r(k-1)v}{(v-1)k} = \frac{b(k-1)}{v-1}.$$

This proves the E-optimality of d^*. □

The next result shows the universal optimality of balanced block designs which were defined in Chapter 3 (Definition 3.5.1).

Theorem 6.3.2 *A balanced block design in $\mathcal{D}(v,b,k), v \geq 3$, whenever existent, is universally optimal over $\mathcal{D}(v,b,k)$.*

Proof. Let $d \in \mathcal{D}(v,b,k)$ be arbitrary. As before, let r_{di} be the replication of the ith treatment in d and $N_d = (n_{dij})$ be the incidence matrix of d. Then as observed earlier,

$$\operatorname{tr}(C_d) = \sum_{i=1}^{v} r_{di} - k^{-1}\sum_{i=1}^{v}\sum_{j=1}^{b} n_{dij}^2 = bk - k^{-1}\sum_{j=1}^{b}\sum_{i=1}^{v} n_{dij}^2.$$

Invoking Lemma 6.2.2, it can be seen that $\operatorname{tr}(C_d)$ is maximized if and only if

$$n_{dij} = [k/v] \text{ or } n_{dij} = [k/v] + 1, \text{ for all } i, j, \tag{6.3.6}$$

where as before, $[x]$ denotes the integral part of x, i.e., if and only if d is a generalized binary design, defined in Section 3.5. Furthermore, it is easy to see that C_d is completely symmetric if

$$\sum_{j=1}^{b} n_{dij} n_{di'j} \text{ is a constant for all } 1 \le i \ne i' \le v, \tag{6.3.7}$$

which, by virtue of Theorem 3.5.1 implies that

$$r_{d1} = r_{d2} = \cdots = r_{dv}. \tag{6.3.8}$$

From Definition 3.5.1, it is seen that the conditions (6.3.6)–(6.3.8) are satisfied by a balanced block design. The universal optimality of balanced block design now follows by invoking Theorem 6.2.1. □

When $k < v$, a balanced block design reduces to a BIB design. Thus, when $k < v$, the universal optimality (and hence, the M-optimality) of BIB designs follows from Theorem 6.3.2. Note that Theorem 6.3.1 also follows from Theorem 6.3.2. However, the proof of Theorem 6.3.1 as given earlier is instructive as it is based on rather elementary considerations.

6.3.2 Optimality of Asymmetric Designs

In this subsection, we present some important results on the optimality of asymmetric designs. We first consider the E-criterion, which is perhaps the simplest to handle. A useful technique due to Takeuchi (1961, 1963), for proving E-optimality is described now. Consider an arbitrary design $d \in \mathcal{D}(v, b, k)$ and let C_d be the C-matrix of d. Also, as before, let $0 = \lambda_{d0} < \lambda_{d1} \le \lambda_{d2} \le \cdots \le \lambda_{d,v-1}$ be the eigenvalues of C_d.

Lemma 6.3.1 *Let E_d be a $v \times v$ matrix defined as*

$$E_d = kC_d - xI_v + yJ_v,$$

where x and y are numbers satisfying $-x + vy > 0$. If E_d is not positive definite (p.d.), then $\lambda_{d1} \le x/k$.

Proof. Let $\theta_1, \ldots, \theta_v$ be the eigenvalues of E_d. Then, it is easy to see that

$$\theta_1 = k\lambda_{d0} - x + yv = -x + yv > 0, \ \theta_2 = k\lambda_{d1} - x, \ldots, \theta_v = k\lambda_{d,v-1} - x.$$

Since E_d is not p.d., $\min_i \theta_i \leq 0 \Rightarrow k\lambda_{d1} - x \leq 0$. □

The above technique is quite useful in obtaining bounds on λ_{d1} and consequently, determining E-optimal designs. Several useful bounds have been obtained by Cheng (1980) and Jacroux (1980) using the above method. Similar bounds were also obtained by Constantine (1981) using a different approach.

Application of the above result can be made in showing the E-optimality of some group divisible designs. As in Chapter 4, the parameters of a group divisible design are denoted by $v = mn, b, r, k, \lambda_1, \lambda_2$ where there are m groups of n treatments each. The next result is due to Takeuchi (1961), which apparently is the first result on the optimality of asymmetric designs.

Theorem 6.3.3 *A group divisible design with $\lambda_2 = \lambda_1 + 1$ is E-optimal over $\mathcal{D}(v, b, k)$.*

Proof. We follow the proof given by Cheng (1996). Let d^* be a group divisible design with m groups of n treatments each, such that $\lambda_2 = \lambda_1 + 1$. Without loss of generality, suppose the lth group in d^* consists of treatment labels $n(l-1)+1, n(l-1)+2, \ldots, nl$, $1 \leq l \leq m$, and suppose the incidence matrix of d^* is N_{d^*}. It can then be seen that (see e.g., Bose and Connor (1952))

$$N_{d^*}N'_{d^*} = I_m \otimes A + (J_m - I_m) \otimes B,$$

where A and B are $n \times n$ matrices given by

$$A = (r - \lambda_1)I_n + \lambda_1 J_n, \quad B = \lambda_2 J_n,$$

and $r = bk/v$ is the common replication number of d^*. For an arbitrary design $d \in \mathcal{D}(v, b, k)$, now define

$$F_d = kC_d - \{(k-1)r + \lambda_1\}I_v + (\lambda_1 + 1)J_v.$$

Clearly, then

$$F_{d^*} = I_m \otimes J_n.$$

The matrix F_{d^*} above is nonnegative definite, has positive row sums and all its diagonal entries are equal to 1.

We now show that F_d is not positive definite for any $d \in \mathcal{D}(v, b, k)$. First observe that $\mathrm{tr}(F_d) \leq \mathrm{tr}(F_{d^*}) = v$, because d^* is binary.

Since each element of F_d, and hence each diagonal element of F_d, is an integer and F_d is $v \times v$, it follows that at least one of the following must hold:

(i) F_d has a diagonal element which is not positive;
(ii) each diagonal element of F_d equals 1.

If (i) holds, then F_d is not positive definite. If (ii) holds then, noting that each row sum of F_d equals $n(\geq 2)$ upon simplification, it follows that at least one off-diagonal entry must be greater than or equal to 1. Then also F_d is not positive definite.

The proof is completed by invoking Lemma 6.3.1 and noting that the smallest positive eigenvalue of C_{d^*} is $k^{-1}\{r(k-1) + \lambda_1\}$. □

Using a similar technique, another result on the E-optimality of group divisible designs was obtained by Cheng (1980) and is stated below.

Theorem 6.3.4 *A group divisible design d^* with $n = 2$ and $\lambda_2 = \lambda_1 - 1 > 0$ is E-optimal in $\mathcal{D}(v, b, k)$.*

Remark 6.3.1 Jacroux (1984a) proved that a group divisible design with groups of size two, $\lambda_2 = \lambda_1 - 1 > 0$ and $k \geq 3$ is also D-optimal.

Remark 6.3.2 Suppose one starts planning an experiment with an optimal incomplete block design and before the actual start of the experiment realizes that there are some more experimental units available, giving rise to the possibility of having one or more additional blocks. Similarly, it is possible in some situations that the experimenter has fewer blocks at her/his disposal than what is required by a (known) optimal design. This kind of situation then calls for results on the optimality of a design which is obtained by augmenting blocks to (or, deleting blocks from) an optimal design. Constantine (1981) showed that when a BIB design or a group divisible design with $\lambda_2 = \lambda_1 + 1$ is extended by the addition of certain disjoint and binary blocks, the resulting design is E-optimal over the entire class of competing designs. When certain disjoint blocks are removed from a BIB design, then also the truncated design is E-optimal. Sathe and Bapat (1985) showed that if some blocks (not necessarily disjoint) are deleted from a BIB design, then under some conditions on the parameters of the BIB design, the resulting design is E-optimal. For some more results on the E-optimality of truncated block designs, see Srivastav and Morgan (2002).

The optimality of asymmetric designs was considered in greater detail by Cheng (1978). One of the major results of Cheng (1978) is stated below.

Theorem 6.3.5 *Suppose there exists a design d^* whose information matrix C_{d^*} has two distinct eigenvalues (both positive) and the larger one has multiplicity one. If d^* maximizes* $\mathrm{tr}(C_d)$ *and maximizes*

$$\mathrm{tr}(C_d) - \left\{\frac{v-1}{v-2}\right\}^{1/2}\left[\mathrm{tr}(C_d^2) - \frac{1}{v-1}(\mathrm{tr}(C_d))^2\right]^{1/2} \qquad (6.3.9)$$

over $\mathcal{D}$, then it is optimal over $\mathcal{D}$ with respect to every type I criteria ψ_f such that $f(0) = \lim_{x\to 0^+} f(x) = \infty$.

Corollary 6.3.1 *Suppose* $\mathrm{tr}(C_d)$ *is a constant for all designs in $\mathcal{D}$. Let there exist a design d^* whose information matrix C_{d^*} has two distinct eigenvalues (both positive) and the larger one has multiplicity one. If d^* minimizes* $\mathrm{tr}(C_d^2)$ *over $\mathcal{D}$, then it is ψ_f-optimal over $\mathcal{D}$ for any convex f such that f' is strictly concave and $f(0) = \lim_{x\to 0^+} f(x) = \infty$.*

Note that in Corollary 6.3.1, f is not required to be nonincreasing as, $\mathrm{tr}(C_d)$ is a constant.

As an application of Theorem 6.3.5, consider a group divisible design d^* with usual parameters $v = mn, b, r, k, \lambda_1, \lambda_2$ such that $m = 2$ and $\lambda_2 = \lambda_1 + 1$. Then, as observed in Chapter 4, C_{d^*} has two distinct nonzero eigenvalues and the larger one of these has multiplicity one. Also, it can be shown that d^* maximizes (6.3.9) over $\mathcal{D}(v, b, k)$. Hence we obtain the following result due to Cheng (1978).

Theorem 6.3.6 *A group divisible design with parameters $v = 2n, b, r, k, \lambda_1, \lambda_2 = \lambda_1 + 1$ is optimal over $\mathcal{D}(v, b, k)$ with respect to every type I criteria ψ_f such that $f(0) = \lim_{x\to 0^+} f(x) = \infty$.*

Remark 6.3.3 Consider the class of designs $\mathcal{D}(v, b, k)$ and suppose $r = [bk/v]$. Consider the set up $v \equiv 2 \pmod 3$, $k = 3, bk = vr + 2$ such that $r(k-1)/(v-1) = \lambda$, say, is an integer. In such a set up, consider a design d^* for which

$$r_{d^*1} = r_{d^*2} = r + 1, r_{d^*i} = r, 3 \le i \le v, \lambda_{d^*12} = \lambda + 2,$$

$$\lambda_{d^*ij} = \lambda \text{ for all other pairs } (i, j),$$

where, as is customary, λ_{dij} is the number of times the ith and jth treatments occur together in the same block in d. One can check that d^* is type I optimal in the general class and thus is the first example of an unequally replicated type I optimal design. Roy and Shah (1984), while searching for a minimal covering design found a d^* with $\lambda = 1$ and $v \equiv 5 \pmod 6$ (a block design is called a covering design if each pair of treatments appears together in at least one block. A covering design with minimum number of blocks is called a minimal covering design). Morgan and Srivastava (2000) gave another family of designs of the type d^* with $v \equiv 2 \pmod 3), \lambda = 2$.

We now introduce *regular graph* designs. The following definition is due to John and Mitchell (1977).

Definition 6.3.1 *A binary, equireplicate and proper block design is called a regular graph design if* $\lambda_{dij} = \lambda$ or $\lambda + 1$ *for some integer* λ.

Consider the class $\mathcal{D}(v, b, k)$ defined earlier and suppose $k < v$. When $k < v$, $\text{tr}(C_d)$ is maximized by binary designs. For a binary design d, the ith diagonal element of C_d is equal to $(k-1)r_{di}/k$ and the (i, j)th off-diagonal element equals $-\lambda_{dij}/k$. Since $\sum_{i=1}^{v} r_{di}$ and $\sum_{i \neq j} \lambda_{dij}$ are constants, $\text{tr}(C_d^2)$ is minimized if

$$r_{d1} = \cdots = r_{dv}, \tag{6.3.10}$$

and

$$\lambda_{dij} = \lambda \text{ or } \lambda + 1 \text{ for some } \lambda. \tag{6.3.11}$$

This analysis proves the (M, S)-optimality of regular graph designs.

Remark 6.3.4 Regular graph designs are expected to have high efficiencies with respect to other more meaningful optimality criteria and in fact, John and Mitchell (1977) conjectured that regular graph designs are also A-, D- and E-optimal. However, counter-examples to this conjecture have been found. For example, Jones and Eccleston (1980), through a computer search, found designs that are not equireplicate and are A-better than the best regular graph design for $(v, b, k) = (10, 10, 2), (11, 11, 2)$, and $(12, 12, 2)$. In contrast to this, Cheng (1992) showed that any (M, S)-optimal design (such as a regular graph design) is A- and D-better than any non-(M, S)-optimal design provided the number of blocks is sufficiently large. For results on the E-optimality of regular graph designs, see Jacroux (1980). The A-optimality of several regular graph designs was established by Jacroux (1985).

We next have the following result (see Cheng (1981), Cheng and Bailey (1991)), which is a modification of Corollary 6.3.1 and is applicable to cases where the information matrix has two distinct positive eigenvalues, but the larger one does not have multiplicity one.

Theorem 6.3.7 *Suppose* $\mathrm{tr}(C_d)$ *is a constant for all designs in* $\mathcal{D}$. *If there exists a design* $d^* \in \mathcal{D}$ *such that* C_{d^*} *has two distinct positive eigenvalues and* d^* *(i) minimizes* $\mathrm{tr}(C_d^2)$ *and, (ii) maximizes the maximum eigenvalue of* C_d *over* $\mathcal{D}$, *then it is* ψ_f*-optimal over* $\mathcal{D}$ *for every convex* f *such that* f' *is strictly concave and* $f(0) = \lim_{x \to 0^+} f(x) = \infty$.

As an application of Theorem 6.3.7, we consider the optimality of a class of designs called *strongly regular graph* designs. A regular graph design is called a strongly regular graph design if it is also a PBIB design with two associate classes and $\lambda_2 = \lambda_1 + 1$ or $\lambda_1 - 1$. Let $\mathcal{D}_1$ be the subclass of $\mathcal{D}(v, b, k)$ consisting of only equireplicate, binary designs with common replication number r. Suppose $\mathcal{D}_1$ contains a strongly regular graph design d^* whose incidence matrix N_{d^*} is such that its concurrence matrix $N_{d^*}N'_{d^*}$ is *singular*. For any design $d \in \mathcal{D}_1$, the largest eigenvalue of C_d is at most r. It follows then that d^* maximizes the largest eigenvalue of C_d, $d \in \mathcal{D}_1$. Since all the designs in $\mathcal{D}_1$ are binary, $\mathrm{tr}(C_d)$ is a constant. Furthermore, since d^* is a regular graph design, it minimizes $\mathrm{tr}(C_d^2)$ over $\mathcal{D}_1$. Invoking Theorem 6.3.7 now, we have the following result due to Cheng and Bailey (1991).

Theorem 6.3.8 *A strongly regular graph design with a singular concurrence matrix is* ψ_f*-optimal over the class of equireplicate binary designs for any convex* f *such that* f' *is strictly concave and* $f(0) = \lim_{x \to 0^+} f(x) = \infty$.

There are several strongly regular graph designs with a singular concurrence matrix. These include the following designs:

(i) All designs satisfying $b < v$ and $\lambda_2 = \lambda_1 \pm 1$;
(ii) all resolvable designs satisfying $b < v + r - 1$ and $\lambda_2 = \lambda_1 \pm 1$;
(iii) all designs based on partial geometries;
(iv) all singular group divisible designs satisfying $\lambda_2 = \lambda_1 - 1$;

(v) all semi-regular group divisible designs satisfying $\lambda_2 = \lambda_1 + 1$.

For some more results on the type I optimality of strongly regular graph designs over the class of binary designs, see Bagchi and Bagchi (2001). We also state the following result due to Yeh (1988) on universally optimal designs in the binary class.

Theorem 6.3.9 *Let $\mathcal{D}_1$ be the class of all binary designs with $v \geq 3$ treatments, $b \geq 2$ blocks and block size $k = v - 1$. Write $b = vm + n$, where m, n are integers, $m > 0$ and $1 \leq n < v$. Suppose d^* is a design whose first vm blocks are obtained by taking m copies of a BIB design involving v treatments and v blocks of size $v-1$ each and the remaining n blocks consist of any n distinct blocks of size $v - 1$ each. Then d^* is universally optimal over $\mathcal{D}_1$.*

We now present some results on the optimality of the dual design of an incomplete block design. Let d be an incomplete block design with v treatments, b blocks each of size k and incidence matrix N_d. Recall from Chapter 4 that the dual of d, say $\bar{d}$, is a block design involving b treatments, v blocks and incidence matrix $N_{\bar{d}} = N_d'$. Let $\mathcal{D}_0(v, b, k)$ be the class of all *equireplicate* block designs with v treatments and b blocks, each of size k. Then, $\bar{d} \in \mathcal{D}_0(b, v, r)$, where $r = bk/v$. It was shown by Shah, Raghavarao and Khatri (1976) that if d is A- (D- or E-) optimal over $\mathcal{D}_0(v, b, k)$, then $\bar{d}$ is A- (D- or E-) optimal over $\mathcal{D}_0(b, v, r)$. A more general result in this direction is due to Eccleston and Kiefer (1981) which we describe below.

The information matrices of d and $\bar{d}$ are respectively given by

$$\begin{aligned} C_d &= rI_v - k^{-1}N_dN_d', \\ C_{\bar{d}} &= kI_b - r^{-1}N_d'N_d. \end{aligned}$$

Since in what follows, the treatments and blocks play symmetric roles, we might assume that $v \leq b$. Let the eigenvalues of N_dN_d' be $\alpha_1, \ldots, \alpha_v$. Then v of the eigenvalues of $N_d'N_d$ are $\alpha_1, \ldots, \alpha_v$ and the remaining $(b - v)$ eigenvalues are each equal to zero. Let λ_{di}, $1 \leq i \leq v$, be the eigenvalues of C_d. Then, $\lambda_{di} = r - \alpha_i/k$, $1 \leq i \leq v$. It follows then that v of the eigenvalues of $C_{\bar{d}}$ are $k\lambda_{d1}/r, k\lambda_{d2}/r, \ldots, k\lambda_{dv}/r$ and the remaining eigenvalues are each equal to k. From this it is clear that d is E-optimal over $\mathcal{D}_0(v, b, k)$ if and only if $\bar{d}$ is E-optimal over $\mathcal{D}_0(b, v, r)$. In particular, the linked block designs (which are duals of BIB designs), the duals of group divisible designs with groups of size two and $\lambda_2 = \lambda_1 - 1 > 0$ and duals of group divisible designs with $\lambda_2 = \lambda_1 + 1$ are E-optimal over the equireplicate class.

Suppose the function f satisfies the following condition:

$$\text{for all } c \text{ there exist } a_c > 0 \text{ and } b_c, \text{ such that } f(cx) = a_c f(x) + b_c. \tag{6.3.12}$$

Then it can be seen that for such an f, d is ψ_f-optimal over $\mathcal{D}_0(v, b, k)$ if and only if $\bar{d}$ is ψ_f-optimal over $\mathcal{D}_0(b, v, r)$. Observe that the A- and D-criteria are covered under such an f. We therefore infer that in general, the duals of optimal designs that are equireplicate are also optimal over the equireplicate class; see also Jacroux (1980) in this connection.

Cheng (1980) proved that linked block designs are E-optimal in the unrestricted class $\mathcal{D}(v, b, k)$; see also Jacroux (1980). Cheng (1990) subsequently showed that the linked block designs are D-optimal without the restriction of equal replication; this result was rediscovered by Pohl (1992). Stronger results on the optimality of dual designs were obtained by Bagchi and Bagchi (2001), and these are stated below. We let $\mathcal{D}_r(v, b, k) \subset \mathcal{D}(v, b, k)$ to denote the class of all equireplicate designs in $\mathcal{D}(v, b, k)$, where $r = bk/v$ is the common replication number.

Theorem 6.3.10 *If $d^* \in \mathcal{D}_r(v, b, k)$ is M-optimal over $\mathcal{D}_r(v, b, k)$, then its dual design is M-optimal over $\mathcal{D}_k(b, v, r)$.*

Since a BIB design is M-optimal and the dual of a BIB design is a linked block design, the following corollary is immediate.

Corollary 6.3.2 *Any linked block design d^* is M-optimal over the class $\mathcal{D}_k(b, v, r)$.*

Theorem 6.3.11 *Any linked block design with parameters $v = s^2 + s$ treatments and $b = s^2$ blocks each of size $k = s^2 - 1$, $s \geq 3$ (which is the complement of the dual of any affine plane of order s – recall (3.4.14)) is M-optimal in the general class $\mathcal{D}(v, b, k)$.*

The next result establishes the A-optimality of the duals of certain BIB designs in the general class. Let d_0 be a BIB design with v_0 treatments and b_0 blocks, each of size k_0, such that $v_0 \geq (k_0^2 + 4)/2$. We then have the following result.

Theorem 6.3.12 *The dual of the complement of d_0 is A-optimal over $\mathcal{D}(v, b, k)$ where $v = b_0, b = v_0$ and $k = b_0 - r_0$.*

For the proofs of the above results, we refer to Bagchi and Bagchi (2001), where several related results can also be found.

6.3.3 D-optimality of Some Incomplete Block Designs

In this subsection, we present some results on the D-optimality of incomplete block designs with small number of experimental units. A graph-theoretic formulation of D-optimality criterion is the main tool that will be used. Such a formulation has been used by several authors including Cheng (1981), Gaffke (1982), Bapat and Dey (1991) and Balasubramanian and Dey (1996). We begin with an elementary result.

Lemma 6.3.2 *Let $A = (a_{ij})$ be a symmetric, nonnegative definite matrix of order n such that (i) $A\mathbf{1}_n = \mathbf{0}$ and (ii) $Rank(A) = n-1$. Then,*
(a) all cofactors of A are equal and positive, and
(b) $\prod_{i=2}^{n} \lambda_i = nCo(A)$,
where $\lambda_1 = 0, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A and $Co(A)$ denotes the common positive cofactor of A.

Proof. (a) Since A is n.n.d., all principal minors of A are nonnegative. If possible, let the principal minor with row and column indices $i_1, i_2, \ldots, i_m$ $(1 \le m \le n-1)$, be zero. Then, there exists a non-null vector $(x_{i_1}, \ldots, x_{i_m})'$ such that

$$\begin{bmatrix} a_{i_1 i_1} & a_{i_1 i_2} & \cdots & a_{i_1 i_m} \\ \vdots & & & \\ a_{i_m i_1} & a_{i_m i_2} & \cdots & a_{i_m i_m} \end{bmatrix} \begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_m} \end{pmatrix} = \mathbf{0}\,.$$

Let x be an $n \times 1$ vector with zero entries everywhere *except* at positions $i_1, \ldots, i_m$, which are occupied by $x_{i_1}, \ldots, x_{i_m}$, respectively. It is then easy to verify that

$$x'Ax = 0 \Rightarrow Ax = \mathbf{0} \Rightarrow \dim(\mathcal{C}(A)) = Rank(A) \le n-2,$$

where dim stands for the dimension of a vector space. This leads to a contradiction as by the hypothesis, $Rank(A) = n-1$. Hence all principal minors of orders $1, 2, \ldots, n-1$ of A are positive. In particular, all cofactors of A are positive.

Let A_{ij} be the cofactor of the element a_{ij} of A. Define $A^* = (A_{ji})$. Then, $AA^* = \mathbf{0}$ which implies that the columns of A^* are proportional to $\mathbf{1}_n$, the only vector in $\mathcal{N}(A)$, the null space of A. By symmetry of A^*, the constants of proportionality are all equal, which implies that all cofactors are equal, proving part (a) of the result.

(b) Note that

$$\begin{aligned}\det(A-\lambda I_n) &= (-\lambda)^n+(-\lambda)^{n-1}\mathrm{tr}_1(A)+(-\lambda)^{n-2}\mathrm{tr}_2(A)\\ &\quad +\cdots+(-\lambda)\mathrm{tr}_{n-1}(A)+\mathrm{tr}_n(A),\end{aligned} \tag{6.3.13}$$

where for $1 \le i \le n$, $\mathrm{tr}_i(A)$ is the sum of the principal minors of order i. Also, if $\lambda_1 = 0, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, then we have

$$\begin{aligned}\det(A-\lambda I_n) &= (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)\\ &= (-\lambda)^n+(-\lambda)^{n-1}\sum_i \lambda_i+(-\lambda)^{n-2}\sum \lambda_i\lambda_j\\ &\quad +\cdots+\prod_i \lambda_i.\end{aligned} \tag{6.3.14}$$

Equating the like powers of λ on the right sides of (6.3.13) and (6.3.14), we have in particular, $\mathrm{tr}_1(A) = \sum \lambda_i = \mathrm{tr}(A), \prod \lambda_i = \mathrm{tr}_n(A) = \det(A)$. Now, the sum of products of λ_i's taken $(n-1)$ at a time $=\mathrm{tr}_{n-1}(A)=$sum of the principal minors of order $n-1 = nCo(A)$. But, in the products of λ_i's taken $n-1$ at a time, $n-1$ are equal to zero, as each of these contain $\lambda_1 = 0$. Hence $\prod_{i=2}^{n} \lambda_i = nCo(A)$. □

We now briefly state some basic notions in graph theory. For a more detailed exposition of graph theory, one may refer to e.g., Harary (1990) or West (2002). A graph G is a pair (V, E) where V is the vertex set and E, the edge set. Two vertices $v_1, v_2 \in V$ are said to be adjacent if there is an edge in E joining v_1 and v_2. An edge is called a *loop* if it connects a vertex with itself. A *multigraph* is one in which more than one edge joins the same pair of vertices. The *degree* of a vertex is the number of edges through that vertex. A *walk* in a graph G is an alternating sequence of vertices and edges $v_0, e_1, v_1, \ldots, e_n, v_n$, beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it. A walk is called closed if $v_0 = v_n$ and is open, otherwise. A walk is called a *path* if all the vertices (and, consequently, all the edges) are distinct. A closed walk is called a *cycle* if its n vertices are distinct and $n \ge 3$. A graph is *connected* if every pair of vertices is joined by a path. A *bipartite* graph G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex in V_1 with a vertex in V_2.

A simple graph (i.e., a graph with no loops or multiple edges) T with p vertices is called a *tree* if it is connected and has precisely $p-1$ edges.

For a multigraph G, a *spanning tree* of G is defined to be a tree, which is a subgraph of G and has the same number of vertices as in G.

For an arbitrary multigraph G with p vertices, let $X(G) = (x_{ij})$ be a $p \times p$ matrix defined as

$$\begin{aligned} x_{ii} &= \text{degree of vertex } i \\ x_{ij} &= -(\text{number of edges joining vertices } i \text{ and } j),\ 1 \leq i \neq j \leq p. \end{aligned}$$

The matrix $X(G)$ defined above is called the *Laplacian* matrix of G. It is known that if G is connected then, $X(G)$ is a symmetric, n.n.d. matrix with zero row sums. The number of spanning trees of a connected multigraph G is called the *complexity* of G, denoted by $c(G)$. It is known that $c(G) = Co(X(G))$, where $Co(X(G))$ is the common positive cofactor of $X(G)$ (*vide* Lemma 6.3.2).

Next, we build a connection between a block design and a bipartite graph. Consider a block design d with v treatments and b blocks each of size $k \geq 2$. Let the treatment labels of d be $1, 2, \ldots, v$ and the block labels, $B_1, B_2, \ldots, B_b$. Any such block design can be described by a bipartite multigraph H_d with vertices labeled as $1, \ldots, v, B_1, \ldots B_b$. A pair of vertices (i, B_j) are joined by n_{dij} parallel edges, where $N_d = (n_{dij})$ is the incidence matrix of d.

As before, let $\mathcal{D}(v, b, k)$ denote the class of all connected block designs with v treatments and b blocks, each of size $k \geq 2$. For a design $d \in \mathcal{D}(v, b, k)$, let H_d be the bipartite multigraph associated with d. The Laplacian matrix of H_d, following the notation of Chapter 2, is then easily seen to be

$$X(H_d) = \begin{pmatrix} R_d & -N_d \\ -N_d' & K_d \end{pmatrix}. \tag{6.3.15}$$

Since d is connected, by Lemma 6.3.2, all cofactors of $C_d = R_d - k^{-1} N_d N_d'$ are equal and positive and $\prod_{i=1}^{v-1} \lambda_{di} = vCo(C_d)$, where $0 = \lambda_{d0} < \lambda_{d1} \leq \lambda_{d2} \leq \cdots \leq \lambda_{d,v-1}$ are the eigenvalues of C_d. We now have the following result, which can be proved by invoking Lemma 6.3.2.

Lemma 6.3.3 *Let d be a block design in $\mathcal{D}(v, b, k)$. Then,*

$$c(H_d) = Co(X(H_d)) = k^b Co(C_d). \tag{6.3.16}$$

Combining Lemma 6.3.2 and Lemma 6.3.3, we thus have

$$\prod_{i=1}^{v-1} \lambda_{di} = vCo(C_d) = (v/k^b) c(H_d), \tag{6.3.17}$$

$c(H_d)$ being the complexity of the graph H_d associated with the design d. We therefore have the following result.

Theorem 6.3.13 *A design $d^* \in \mathcal{D}(v, b, k)$ is D-optimal over $\mathcal{D}(v, b, k)$ if and only if the bipartite multigraph H_{d^*} associated with d^* has the maximum number of spanning trees, i.e., if and only if*

$$c(H_{d^*}) \geq c(H_d) \text{ for any other design } d \in \mathcal{D}(v, b, k).$$

As an application of Theorem 6.3.13, consider a block design with v treatments and b blocks each of size k. A necessary condition for the design to be connected is that $bk \geq b + v - 1$. Designs that are connected and for which the number of experimental units bk attains the lower bound are called *minimally connected*. Let $\mathcal{D}_0(v, b, k)$ be the class of all minimally connected designs. It can be seen that all designs in $\mathcal{D}_0(v, b, k)$ are necessarily binary. The bipartite graph H_d for any $d \in \mathcal{D}_0(v, b, k)$ is itself a tree and thus has just one spanning tree. This means that $c(H_d) = 1$ for all designs $d \in \mathcal{D}_0(v, b, k)$, leading to the following result due to Bapat and Dey (1991).

Theorem 6.3.14 *All minimally connected designs are equivalent according to the D-optimality criterion.*

Remark 6.3.5 The result in Theorem 6.3.14 is somewhat disappointing in the sense that the D-criterion is unable to discriminate among designs in $\mathcal{D}_0(v, b, k)$. However, if one considers other criteria like the E- or A-optimality criteria, then as described below, it is possible to find a unique design that is optimal according to these criteria.

Suppose $d^* \in \mathcal{D}_0(v, b, k)$ is constructed as follows: Label the treatments $0, 1, \ldots, v-1$. Distribute the $v-1 = b(k-1)$ treatments $1, 2, \ldots, v-1$ at the rate of $k-1$ treatments per block over the b blocks and then add treatment 0 to each of the blocks. Then, d^* is uniquely A- and E-optimal over $\mathcal{D}_0(v, b, k)$ (see Bapat and Dey (1991) and Mandal, Shah and Sinha (1991) for details). A result similar to that in Theorem 6.3.14 in a slightly different context and using different tools was obtained by Mukerjee, Chatterjee and Sen (1986) and also by Krafft (1990).

Consider now the class $\mathcal{D}_1(v, b, k)$ of all connected designs with v treatments and b blocks each of size k, where the parameters satisfy the condition $bk = b + v$. The bipartite graph associated with any design in $\mathcal{D}_1(v, b, k)$ has precisely one more edge than in the graph associated with a design in $\mathcal{D}_0(v, b, k)$. Recall that the graph associated with any design in $\mathcal{D}_0(v, b, k)$ is a tree. The consequence of adding an edge to a tree is that now we have exactly one cycle, *provided* the extra edge is not a multiple edge (observe that a tree is a connected graph with no cycles). But, if there is a multiple edge, the number of spanning trees is exactly *two*. Note that the bipartite graph will have a multiple edge if and only if the associated design is non-binary.

Since the length of a cycle is *at least* three and the number of spanning trees is precisely the length of the cycle, it follows that a non-binary design cannot be D-optimal over $\mathcal{D}_1(v, b, k)$. We may therefore restrict the search for a D-optimal design in $\mathcal{D}_1(v, b, k)$ to only binary designs. The graphs associated with such designs will have precisely one cycle. Since the graph is bipartite, the length of the cycle cannot exceed $2\min(b, v) = 2b$, as $b(k-1) = v$. Thus, for any design $d \in \mathcal{D}_1(v, b, k)$,

$$c(H_d) \leq 2b. \tag{6.3.18}$$

Consider now the design d^*, with treatment labels $1, \ldots, v = b(k-1)$ and the blocks given by

$$(1, 2, \ldots, k); (k, k+1, \ldots, 2k-1); (2k-1, 2k, \ldots, 3k-2);$$

$$\cdots$$

$$[(b-2)(k-1), (b-2)(k-1)+2, \ldots, (b-1)(k-1)+1];$$
$$[(b-1)(k-1)+1, (b-1)(k-1)+2, \ldots, b(k-1), 1].$$

The length of the cycle in the graph associated with the above design is $2b$ and hence $c(H_{d^*}) = 2b$. We thus have the following result due to Balasubramanian and Dey (1996).

Theorem 6.3.15 *The design d^* as described above is D-optimal over $\mathcal{D}_1(v, b, k)$.*

For more results on D-optimality of block designs using the above graph-theoretic formulation, see Gaffke (1982), Balasubramanian and Dey (1996) and Dey, Shah and Das (1995).

6.4 Optimal Designs for Test-Control Comparisons

In Section 5.4 (Chapter 5), we have described some incomplete block designs suitable for control-test treatments comparisons. The issue of optimality of such designs is considered now.

As in Section 5.4, we consider the situation where there are v test treatments which are to be compared with a control treatment. The (fixed) effects of the test treatments are denoted by $\tau_1, \ldots, \tau_v$ and that of the control treatment by τ_0. The contrasts of primary interest are $\tau_i - \tau_0, 1 \leq i \leq v$. Under a block design d, let $\hat{\tau}_{di} - \hat{\tau}_{d0}$ denote the BLUE of $\tau_i - \tau_0$. In this set up, a design d^* in a certain class of competing designs $\mathcal{D}$ is said to be A-optimal over $\mathcal{D}$ if

$$\sum_{i=1}^{v} \mathrm{Var}(\hat{\tau}_{d^*i} - \hat{\tau}_{d^*0}) \leq \sum_{i=1}^{v} \mathrm{Var}(\hat{\tau}_{di} - \hat{\tau}_{d0}) \text{ for any other design } d \in \mathcal{D}.$$

A design d^* in $\mathcal{D}$ is said to be MV-optimal over $\mathcal{D}$ if

$$\max_{1 \leq i \leq v} \mathrm{Var}(\hat{\tau}_{d^*i} - \hat{\tau}_{d^*0}) \leq \max_{1 \leq i \leq v} \mathrm{Var}(\hat{\tau}_{di} - \hat{\tau}_{d0}) \text{ for any other design } d \in \mathcal{D}.$$

Let $\mathcal{D}(v+1, b, k)$ denote the class of all connected block designs involving v test and a single control treatment and b blocks each of size k. Let d be a typical member of $\mathcal{D}(v+1, b, k)$ and $N = (n_{dij})$ be the $(v+1) \times b$ incidence matrix of d. If $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots, \tau_v)'$ denotes the vector of treatment effects and $L = (-\mathbf{1}_v, I_v)$, then the contrasts of interest can be represented by $L\boldsymbol{\tau}$. Under the usual intra-block model, the information matrix for estimating linear functions of treatment effects, using a design d, is given by $C_d = \mathrm{diag}(r_{d0}, r_{d1}, \ldots, r_{dv}) - k^{-1}N_d N_d'$ where for $0 \leq i \leq v$, r_{di} is the replication of the ith treatment in d. If one partitions C_d as

$$C_d = \begin{pmatrix} c_{d00} & \alpha_d' \\ \alpha_d & M_d \end{pmatrix} \tag{6.4.1}$$

then it has been shown by Bechhofer and Tamhane (1981) that

$$(LC_d^- L')^{-1} = M_d,$$

that is, M_d is the information matrix for the treatment-control contrasts. In view of this, an A-optimal design minimizes $\mathrm{tr}(M_d^{-1})$ over $\mathcal{D}(v+1, b, k)$ and an MV-optimal design minimizes the largest diagonal element of M_d^{-1} over $\mathcal{D}(v+1, b, k)$.

One way to determine an optimal block design is to construct, if possible, an *orthogonal* block design, such that within each block the replication of the treatments are optimal for a zero-way elimination of heterogeneity model (i.e., the completely randomized design model). This result and its generalizations has been used by several authors, e.g., Magda (1980) and Kunert (1983) in a different context. The following result, which is a different version of the result of Magda and Kunert, is due to Majumdar (see Majumdar (1996)). Let $\mathcal{D}$ be a class of competing designs and consider the following two models for the $n \times 1$ observations vector, when the observations are obtained by using a design $d \in \mathcal{D}$:

$$\begin{aligned} \mathcal{M}_1 &: \boldsymbol{Y} = X_{1d}\boldsymbol{\tau} + X_{2d}\boldsymbol{\theta}_2 + \text{error} \\ \mathcal{M}_2 &: \boldsymbol{Y} = X_{1d}\boldsymbol{\tau} + X_{2d}\boldsymbol{\theta}_2 + X_{d3}\boldsymbol{\theta}_3 + \text{error} \end{aligned} \tag{6.4.2}$$

where $\boldsymbol{\tau}$ is a vector of treatment effects and $\boldsymbol{\theta}_2, \boldsymbol{\theta}_3$ are vectors of nuisance parameters. The errors are as usual, uncorrelated random variables with zero means and constant variance σ^2. Kunert calls $\mathcal{M}_2$ *finer* than $\mathcal{M}_1$. Majumdar's result is then as follows.

Theorem 6.4.1 *Suppose a design $d_0 \in \mathcal{D}$ is A- (respectively, MV-) optimal for treatment-control comparisons under the model $\mathcal{M}_1$ and,*

$$X'_{1d_0}X_{3d_0} = X'_{1d_0}X_{2d_0}(X'_{2d_0}X_{2d_0})^{-}X'_{2d_0}X_{3d_0}, \tag{6.4.3}$$

then d_0 is A- (respectively, MV-) optimal for treatment-control contrasts under $\mathcal{M}_2$.

Consider now a zero-way elimination of heterogeneity set up. The model here is

$$Y_{ij} = \mu + \tau_i + \text{error} \tag{6.4.4}$$

where the symbols have their obvious meanings. Then, under this model, it is easy to see that

$$\text{Var}(\hat{\tau}_{di} - \hat{\tau}_{d0}) = \sigma^2(r_{di}^{-1} + r_{d0}^{-1}). \tag{6.4.5}$$

An A-optimal design in this situation is therefore obtained by minimizing

$$\sum_{i=1}^{v}(r_{di}^{-1} + r_{d0}^{-1}), \text{ subject to } \sum_{i=0}^{v} r_{di} = n.$$

Then the following result is not hard to prove.

Theorem 6.4.2 *If v is a perfect square and $n \equiv 0 \pmod{(v + \sqrt{v})}$, then a design d_0 given by*

$$r_{d_01} = \cdots = r_{d_0v} = \frac{n}{v + \sqrt{v}}, \; r_{d_00} = r_{d_01}\sqrt{v} \tag{6.4.6}$$

is A-optimal for treatment-control comparisons under the model (6.4.4).

Combining Theorems 6.4.1 and 6.4.2, we thus have the following result.

Corollary 6.4.1 *Given v, b, k, suppose v is a perfect square, k is a multiple of $(v + \sqrt{v})$ and $d_0 \in \mathcal{D}(v+1, b, k)$ is such that for $1 \leq j \leq b$,*

$$n_{d_0 1j} = \cdots = n_{d_0 vj} = \frac{k}{v + \sqrt{v}}, \; n_{d_0 0j} = n_{d_0 1j}\sqrt{v}. \tag{6.4.7}$$

Then d_0 is A-optimal for treatment-control contrasts under a block design model over $\mathcal{D}(v+1, b, k)$.

The scope of the result in Corollary 6.4.1 is limited as firstly, it requires v to be a perfect square and more importantly, k has to be multiple of $(v + \sqrt{v})$, which is quite large in comparison to v. For example, with $v = 4$, k has to be at least 6 and for $v = 9$, k has to be at least 12. In practice, one often desires to have a design with small block sizes and therefore, we need to look for suitable incomplete block designs for the problem under consideration. We thus restrict our attention to designs for which

$$2 \leq k \leq v. \tag{6.4.8}$$

Suppose $d \in \mathcal{D}(v+1, b, k)$ is arbitrary. An argument, based on consideration of convexity via a technique of averaging (*cf.* Kiefer (1975)) can be employed to show that

$$\operatorname{tr}(LC_d^- L') \geq \operatorname{tr}(LC_{da}^- L'), \tag{6.4.9}$$

where $C_{da} = (1/v!) \sum_Z ZC_d Z'$, the sum being taken over all permutation matrices Z of order $(v+1) \times (v+1)$ that correspond to the permutations of the v test treatments only. Using the partitioned form as in (6.4.1), it can be seen that $M_{da} = (LC_{da}^- L')^{-1}$ is completely symmetric. In general, there may not exist a design $d \in \mathcal{D}(v+1, b, k)$ for which C_{da} is the information matrix for treatments. If there exists such a design, say d_1, then $M_{d_1} = M_{d_1 a}$ is completely symmetric and furthermore, the vector α_{d_1} in (6.4.1) has all its elements equal. This implies that d_1 is a supplemented balanced design.

Based on the preceding discussion, a strategy for obtaining an A-optimal design can be as follows: starting with an arbitrary design $d \in \mathcal{D}(v+1, b, k)$, one can use (6.4.9) to obtain a lower bound for the value of the A-criterion for d. This lower bound can then be minimized over $\mathcal{D}(v+1, b, k)$ and a design that attains the minimum value is the A-optimal design. This strategy was followed by Majumdar and Notz (1983), which we describe now.

For integers v, b, k, x and z, let

$$\begin{aligned} g(x,z) &= kv(v-1)^2\{bkv(k-1)-(bx+z)(vk-v+k) \\ &\quad +(bx^2+2xz+z)\}^{-1} \\ &\quad +vk\{k(bx+z)-(bx^2+2xz+z)\}^{-1}, \end{aligned} \tag{6.4.10}$$

$$G = \{0,1,\ldots,[k/2]-1\} \times \{0,1,\ldots,b\} - \{0,0\}. \tag{6.4.11}$$

For a design d which is a BTIB $(v,b,k;x,z)$, $\operatorname{tr}(LC_d^- L') = g(x,z)$, that is, g gives the value of the A-criterion. In the design d, $r_{d0} = bx+z$. For $(x,z) \in G$, the function g and the set G can be represented in terms of $r = bx+z$ as follows:

$$\begin{aligned} g^*(r) &= kv(v-1)^2\{bkv(k-1)-r(vk-v+k)+h(r)\}^{-1} \\ &\quad +vk\{rk-h(r)\}^{-1}, \end{aligned} \tag{6.4.12}$$

where

$$h(r) = b([r/b])^2 + (2[r/b]+1)(r-b[r/b]), \tag{6.4.13}$$

$$G^* = \{1,\ldots,b[k/2]\}. \tag{6.4.14}$$

The result of Majumdar and Notz (1983) can now be stated.

Theorem 6.4.3 *Let t, s be integers given by*

$$g(t,s) = \min_{(x,z)\in G} g(x,z). \tag{6.4.15}$$

With $2 \le k \le v$, for any design $d \in \mathcal{D}(v+1,b,k)$,

$$\operatorname{tr}(LC_d^- L') \ge g(t,s), \tag{6.4.16}$$

with equality if d is a BTIB $(v,b,k;t,s)$. Therefore, a BTIB $(v,b,k;t,s)$ is A-optimal for treatment-control contrasts over $\mathcal{D}(v+1,b,k)$.

The following result is equivalent to the one in Theorem 6.4.3.

Theorem 6.4.4 *Let r^* be an integer defined by*

$$g^*(r^*) = \min_{r\in G^*} g^*(r). \tag{6.4.17}$$

Then, with $2 \le k \le v$, for any design $d \in \mathcal{D}(v+1,b,k)$,

$$\operatorname{tr}(LC_d^- L') \ge g^*(r^*), \tag{6.4.18}$$

with equality if d is a BTIB $(v,b,k;t,s)$ where $bt+s=r^*$. Therefore, a BTIB $(v,b,k;t,s)$ satisfying $bt+s=r^*$ is A-optimal for treatment-control contrasts over $\mathcal{D}(v+1,b,k)$.

As an application of Theorem 6.4.3, one can see that the BTIB (7,7,4;1,0) design in Example 5.4.2 (i) of Chapter 5 is A-optimal over $\mathcal{D}(8,7,4)$. Similarly, the BTIB (6,18,5;1,6) design given in Example 5.4.2 (ii) is A-optimal over $\mathcal{D}(7,18,5)$.

Based on Theorem 6.4.3 and Theorem 5.4.1 (Chapter 5), Hedayat and Majumdar (1984) suggested a procedure for obtaining optimal designs that consists of the following steps:

Step 1. Starting from v, b, k, determine t and s that minimize $g(x, z)$.

Step 2. Verify conditions of Theorem 5.4.1(i), using t and s from Step 1. If the conditions are not satisfied, then Theorem 6.4.3 cannot be applied to the class $\mathcal{D}(v+1, b, k)$. If the conditions are satisfied, go to Step 3 below.

Step 3. Attempt to construct a BTIB$(v, b, k; t, s)$.

There is of course no guarantee that Step 3 can always be implemented, even if the conditions of Theorem 5.4.1(i) hold. As stated in Chapter 5, the construction of (optimal) R-type designs reduces to the problem of finding BIB designs while the construction of S-type designs is more involved.

For specific values of v, b, k, the minimization in (6.4.15) gives an elegant algebraic solution which can lead to the solution of families of A-optimal designs with nice combinatorial properties. In this context, the following result was obtained by Hedayat and Majumdar (1985).

Theorem 6.4.5 *A BTIB$(v, b, k; 1, 0)$ design is A-optimal for treatment-control contrasts in $\mathcal{D}(v+1, b, k)$ whenever $(k-2)^2 + 1 \leq v \leq (k-1)^2$.*

When $v = (k-2)^2 + 1$, an A-optimal design can be constructed by taking the design d_2 in (5.4.2) (Chapter 5) to be a finite projective plane of order $k-2$. Similarly, when $v = (k-1)^2$, an A-optimal design is obtained by taking the design d_2 in (5.4.2) to be a finite Euclidean plane of order $k-1$.

Theorem 6.4.5 was generalized by Stufken (1987) to the following result.

Theorem 6.4.6 *A BTIB$(v, b, k; t, 0)$ design is A-optimal for treatment-control comparisons over $\mathcal{D}(v+1, b, k)$ whenever $(k-t-1)^2 + 1 \leq t^2 v \leq (k-t)^2$.*

Example 6.4.1 Let s be a prime or a prime power. Then, as observed in Chapter 3, a BIB design d with $v = s^2$ treatments and $b = s^2 + s$ blocks each of size s exists. The complementary design, $\bar{d}$ is a BIB design with v treatments and b blocks each of size $s^2 - s$. Suppose we take $v = s^2, b = s^2 + s$ and $k = s^2 - s + t$. Then, $vt^2 = (k-t)^2$ if and only if $t = s - 1$. With this value of t, $vt^2 - (k - t - 1)^2 - 1 > 0$. Therefore, a BTIB$(s^2, s^2 + s, s^2 - 1; s - 1, 0)$ can be constructed by augmenting each block of $\bar{d}$ by $s - 1$ replicates of the control which is A-optimal. For instance, with $s = 3$, we have an A-optimal BTIB$(9, 12, 8; 2, 0)$.

Remark 6.4.1 Theorems 6.4.5 and 6.4.6 provide sufficient conditions for the A-optimality of reinforced BIB designs of Das (1958).

Cheng, Majumdar, Stufken and Ture (1988) provide an optimal family of S-type designs and their result is given below.

Theorem 6.4.7 *Let $s \geq 3$ be a prime or a prime power and δ be a positive integer. Then there exists a BTIB$(s^2 - 1, \delta(s + 2)(s^2 - 1), s; 0, \delta(s + 1)(s^2 - 1))$. This design is A-optimal for treatment control comparisons in $\mathcal{D}(s^2, \delta(s + 2)(s^2 - 1), s)$.*

For the actual construction of the designs in Theorem 6.4.7, the original source may be consulted.

Remark 6.4.2 Majumdar and Notz (1983) also considered optimality criteria other than the A-criterion for the treatment-control contrasts. Results similar to that in Theorem 6.4.5 were obtained by Giovagnoli and Wynn (1985) using the approximate theory. Additional results on the determination of A-optimal or A-efficient designs for comparing test treatments with control(s) in different contexts have been obtained by Parsad, Gupta and Prasad (1995), Gupta, Pandey and Parsad (1998) and Gupta, Ramana and Parsad (1999, 2002). Some highly A-efficient BTIB designs were presented by Das, Dey, Kageyama and Sinha (2005). For details on these, the original sources may be consulted.

Turning to the problem of finding an MV-optimal design d for test treatment-control contrasts, we first observe the following fact. For a BTIB design,

$$\max_{1 \leq i \leq v} \mathrm{Var}(\hat{\tau}_{di} - \hat{\tau}_{d0}) = v^{-1} \sum_{i=1}^{v} \mathrm{Var}(\hat{\tau}_{di} - \hat{\tau}_{d0}).$$

This clearly implies that an A-optimal BTIB design is also MV-optimal. However, this result cannot be used if the A-optimality of the design is not known. Jacroux (1987) gave a procedure that is often successful in locating an MV-optimal design in such cases. While we do not elaborate on this procedure and refer to Jacroux (1987) for details, we give below an example.

Example 6.4.2 Let $v = 11 = b, k = 6$. Using Theorem 6.4.4 it is seen that $r^* = 14$. However, there is no BTIB$(11, 11, 6; t, s)$ with $11t + s = 14$ (i.e., $t = 1, s = 3$). Jacroux (1987) gave the following BTIB(11,11,6;1,0) design that is MV-optimal over $\mathcal{D}(12, 11, 6)$:

$$(0,2,4,5,6,10); (0,3,5,6,7,11); (0,1,4,6,7,8); (0,2,5,7,8,9);$$
$$(0,3,6,8,9,10); (0,4,7,9,10,11); (0,1,5,8,10,11); (0,1,2,6,9,11);$$
$$(0,1,2,3,7,10); (0,2,3,4,8,11); (0,1,3,4,5,9).$$

Remark 6.4.3 In contrast to the optimal estimation of treatment-control contrasts that we have considered so far in this section, one may also find optimal designs for simultaneous confidence intervals. For a zero-way elimination of heterogeneity set up, the work on optimal designs for simultaneous confidence intervals was initiated by Dunnett (1955) and further studied, among others, by Bechhofer (1969), Bechhofer and Nocturne (1972), Bechhofer and Tamhane (1983) and Spurrier and Nizam (1990). In the block design (or, the one-way elimination of heterogeneity) set up, Bechhofer and Tamhane (1981) were the first to consider the problem of finding optimal designs for simultaneous confidence intervals. Further contributions were made by Notz and Tamhane (1983) and Ture (1982, 1985). For more on these as also on related issues, we refer to the excellent review article by Majumdar (1996), where more references can also be found.

6.5 Optimal Designs for Parallel Line assays

Incomplete block designs for parallel line assays were considered in Section 5.3 (Chapter 5). In this section, we consider the optimality of such incomplete block designs. The work on optimality of incomplete block designs for parallel line assays was initiated by Mukerjee and Gupta (1995) with reference to the A-optimality criterion. Mukerjee (1996) considered the same problem with reference to the D-optimality criterion. Subsequent work in this direction, all with reference to the A-criterion, are due to Chai (2002), Chai and Das (2001), Chai, Das and Dey (2001, 2003) and Srivastava, Parsad, Dey and Gupta (2007, 2008).

Let us first consider the set up of a symmetric parallel line assay involving $m \geq 2$ doses of each of the standard and test preparations. Suppose this assay is conducted using a block design with b blocks of $k < v(= 2m)$ experimental units each and interest lies in the contrasts L_p, L_1 and L_1', defined in Chapter 5 (Section 5.3). The normalized versions of these three contrasts are (say) $g_1'\tau, g_2'\tau$ and $g_3'\tau$, where

$$\begin{aligned} g_1 &= (2m)^{-1/2}(\mathbf{1}_m', -\mathbf{1}_m')', \\ g_2 &= [6/\{m(m^2-1)\}]^{1/2}(e_1', e_1')', \\ g_3 &= [6/\{m(m^2-1)\}]^{1/2}(e_1', -e_1')', \end{aligned} \tag{6.5.1}$$

and e_1 is as defined in (5.3.17). Interest then lies in $G\tau$, where the $3 \times v$ matrix G is given by $G = (g_1, g_2, g_3)'$. It can be seen that the diagonal elements of $G'G$ are $\theta_1, \ldots, \theta_v$, where, for $1 \leq j \leq m$,

$$\theta_j = \theta_{m+j} = \frac{1}{2m} + \frac{12}{m(m^2-1)}\left\{j - \frac{1}{2}(m+1)\right\}^2. \tag{6.5.2}$$

Note that for $1 \leq j \leq m$,

$$\theta_j = \theta_{m+1-j} = \theta_{m+j} = \theta_{2m+1-j}. \tag{6.5.3}$$

Let $\mathcal{D}(v, b, k)$ denote the class of all designs involving v treatments and b blocks each of size $k(< v)$ and let $\mathcal{D}_1(v, b, k)$ be the subclass of $\mathcal{D}(v, b, k)$ consisting only of those designs that keep $G\tau$ estimable. An A-optimal design for $G\tau$ in $\mathcal{D}(v, b, k)$ is one that belongs to $\mathcal{D}_1(v, b, k)$ and minimizes $\mathrm{tr}(\mathbb{D}(G\hat{\tau}))$ over $\mathcal{D}_1(v, b, k)$. Mukerjee and Gupta (1995) suggested the following steps for the construction of an A-optimal design when $m = 2u$ is even and $k \equiv 0 \pmod 4$:

Step 1. Minimize $\sum_{j=1}^{u}(\theta_j/q_j)$ with respect to $q = (q_1, \ldots, q_u)'$ such that the q_i's are positive integers satisfying $q_1 + \cdots + q_u = bk/4$. Let $(q_1^*, \ldots, q_u^*)'$ be a choice of q where this minimum is attained.

Step 2. Construct a design d^* involving u treatments and b blocks each of size $k/4$ such that for $1 \leq j \leq u$, the jth treatment is replicated q_j^* times in d^*.

Step 3. Obtain a design d from d^* by replacing the jth treatment in d^* by the four treatments s_j, s_{m+1-j}, t_j and t_{m+1-j}, where $s_1, \ldots, s_m$ are the doses of the standard treatment and $t_1, \ldots, t_m$ are those of the test treatment.

The design d so constructed is an A-optimal design in $\mathcal{D}(v, b, k)$. For a proof of this fact, see Mukerjee and Gupta (1995). The following example serves as an illustration of the above steps.

Example 6.5.1 Let $m = 6$ and suppose an A-optimal design is desired in $b = 3$ blocks of size $k = 8$ each. Here $v = 2m = 12$. By (6.5.2), we have $\theta_1 = 185/420, \theta_2 = 89/420, \theta_3 = 41/420$. Invoking Step 1 of the construction, it is seen that $(q_1^*, q_2^*, q_3^*)'$ is uniquely given by $q_1^* = 3, q_2^* = 2, q_3^* = 1$. We can thus take d^* as consisting of the three blocks (1,2),(1,2),(1,3). Finally, by Step 3, an A-optimal design is given by

$$d = \begin{array}{l} (s_1, s_6, t_1, t_6, s_2, s_5, t_2, t_5)\,; \\ (s_1, s_6, t_1, t_6, s_2, s_5, t_2, t_5)\,; \\ (s_1, s_6, t_1, t_6, s_3, s_4, t_3, t_4)\,. \end{array}$$

Remark 6.5.1 Adhering to Example 6.5.1, we know by Theorem 5.3.1 that an L-design, say d_1, exists with $v = 12, b = 3, k = 8$. The efficiency of d_1, given by the ratio

$$\mathrm{tr}(\mathbb{D}(G\hat{\tau})_d)/\mathrm{tr}(\mathbb{D}(G\hat{\tau})_{d_1})$$

equals 0.9344. Thus, in this case, there is substantial gain in using the nonequireplicate A-optimal design in preference to a comparable equireplicate L-design. However, the L-designs in general perform well under the A-criterion when the competing class of designs is allowed to include nonequireplicate designs also. Given the integers v, b, k, suppose an L-design d_1 exists. Then, it can be shown that

$$\mathrm{tr}(\mathbb{D}(G\hat{\tau})_{d_1}) = 3v\sigma^2/(bk),$$

where σ^2 is the variance of an observation. On the other hand, it can be shown that for an arbitrary design $d \in \mathcal{D}(v, b, k)$,

$$\mathrm{tr}(\mathbb{D}(G\hat{\tau})_d) \geq (bk)^{-1} \left(\sum_{j=1}^{v} \theta_j^{1/2} \right)^2 \sigma^2,$$

where the θ_j's are as in (6.5.2). Hence, if an L-design exists, its A-efficiency, as a member of $\mathcal{D}(v, b, k)$ is bounded below by

$$E = (3v)^{-1} \left(\sum_{j=1}^{v} \theta_j^{1/2} \right)^2 . \tag{6.5.4}$$

The quantity E in (6.5.4) is always at least as large as 0.91. This fact justifies the use of L-designs in situations where an A-optimal design is not as yet known.

The L-designs, apart from having high A-efficiencies in general are also MV-optimal. Recall that under the MV-criterion, one chooses a design in $\mathcal{D}(v, b, k)$ that keeps $G\boldsymbol{\tau}$ estimable and minimizes $\max\limits_{1\le i\le 3} \mathrm{Var}(\boldsymbol{g}_i'\hat{\boldsymbol{\tau}})$. The following result due to Gupta and Mukerjee (1996) is relevant in this context.

Theorem 6.5.1 *Given v, b and $k < v$, if an L-design exists, then it is MV-optimal for $G\boldsymbol{\tau}$ in $\mathcal{D}(v, b, k)$.*

Proof. By (6.5.1), the absolute value of each element of $\boldsymbol{g}_1$ is $v^{-1/2}$. Hence, for any design that keeps $G\boldsymbol{\tau}$ estimable, by part (i) of Lemma 5.3.1 and the Cauchy-Schwarz inequality,

$$\max_{1\le i\le 3} \mathrm{Var}(\boldsymbol{g}_i'\hat{\boldsymbol{\tau}}) \ge \mathrm{Var}(\boldsymbol{g}_1'\hat{\boldsymbol{\tau}}) \ge (\boldsymbol{g}_1' R^{-1} \boldsymbol{g}_1)\sigma^2 \ge v\sigma^2/(bk). \tag{6.5.5}$$

Again, from part (ii) of Lemma 5.3.1, for an L-design,

$$\mathrm{Var}(\boldsymbol{g}_i'\hat{\boldsymbol{\tau}}) = v\sigma^2/(bk),\ 1 \le i \le 3. \tag{6.5.6}$$

The result now follows by (6.5.5). □

The A-optimal designs for the estimation of $G\boldsymbol{\tau}$ obtained by Mukerjee and Gupta (1995) require a block size that is an integral multiple of four. Moreover, these designs are not always connected. Chai, Das and Dey (2001) obtained incomplete block designs which are A-optimal for only two contrasts, viz., preparation and combined regression contrasts and, in many cases have high A-efficiency for the parallelism contrast as well. Their technique works well for both symmetric and asymmetric assays. The motivation for considering only two of the three contrasts of major importance are: (i) the two contrasts considered are the ones used for the estimation of relative potency, the primary objective of the assay, (ii) consideration of only two contrasts gives greater flexibility in respect of the block size and (iii) it is always possible to obtain a connected design when only two contrasts are considered.

The technique of Chai *et al.* (2001) for obtaining an A-optimal design for the preparation and combined regression contrasts is a modification of the technique of Mukerjee and Gupta (1995). For details on these, the original paper may be consulted.

For symmetric parallel line assays, Chai, Das and Dey (2003) obtained a class of incomplete block designs, called nearly L-designs which are available with odd block sizes (recall that for L-designs, the block size is necessarily even) and have high efficiencies under the A-criterion for all the three contrasts. The basic strategy followed by Chai *et al.* (2003) is to first establish a link between linear trend-free block designs (*cf.* Section 5.6.3) and nearly L-designs. With the help of this connection, necessary and sufficient conditions for the existence of nearly L-designs are obtained and then, a construction method for such designs is provided. Here is an example of such designs.

Example 6.5.2 Let $m = 5, k = 5, b = 12$. Consider the following design d_0 with four distinct blocks, the desired design in $b = 12$ blocks being obtained by repeating each of the distinct blocks thrice:

$$d_0 = \begin{array}{l} (s_2, s_3, s_4, t_1, t_5)\,; \\ (s_1, s_3, s_5, t_2, t_4)\,; \\ (s_1, s_5, t_2, t_3, t_4)\,; \\ (s_2, s_4, t_1, t_3, t_5)\,. \end{array}$$

The A-efficiency of the preparation contrast under this design is at least 0.9921 and the overall efficiency of the design, taking all three contrasts into consideration is at least 0.9973.

For some results on the A-optimality of incomplete block designs for asymmetric parallel line assays, see Chai, Das and Dey (2001).

6.6 Optimal Designs for Diallel Crosses

Incomplete block designs for complete diallel crosses have been considered in Chapter 5 (Section 5.5). In this section, we consider the optimality aspects of such designs. As in Section 5.5, let p be the number of inbred lines and it is desired to conduct a complete diallel cross experiment using an incomplete block design d involving b blocks each of size $k \geq 2$. The number of treatments (crosses) in such an experiment is $p(p-1)/2$. As in Section 5.5, let r_{di} denote the number of times the ith cross appears in d, $1 \leq i \leq p(p-1)/2$ and similarly, let s_{dj} denote the number of times the jth line occurs in d, $1 \leq j \leq p$. Then, under the model (5.5.12), the coefficient matrix of the reduced normal equations for the general combining ability (g.c.a.) effects is given by (5.5.14). A design d is connected if and only if $Rank(C_d) = p - 1$ where C_d is as in (5.5.14). Let $\mathcal{D}(p, b, k)$ denote the class of all such connected designs involving p lines and b blocks each of size k. We then have the following result due to Das, Dey and Dean (1998).

Theorem 6.6.1 *For any design $d \in \mathcal{D}(p, b, k)$,*

$$\operatorname{tr}(C_d) \le k^{-1}b\{2k(k-1-2x) + px(x+1)\}, \tag{6.6.1}$$

where $x = [2k/p]$. Equality holds if and only if $n_{dij} = x$ or $x+1$ for all $i = 1, \ldots, p, j = 1, \ldots, b$.

Proof. For an arbitrary $d \in \mathcal{D}(p, b, k)$,

$$\begin{aligned} \operatorname{tr}(C_d) &= \sum_{i=1}^{p} s_{di} - k^{-1}\sum_{i=1}^{p}\sum_{j=1}^{b} n_{dij}^2 \\ &= 2bk - k^{-1}\sum_{i=1}^{p}\sum_{j=1}^{b} n_{dij}^2. \end{aligned}$$

Since $\sum_{i=1}^{p}\sum_{j=1}^{b} n_{dij} = 2bk$, by Lemma 6.2.2,

$$\sum_{i=1}^{p}\sum_{j=1}^{b} n_{dij}^2 \ge b\{2k(2x+1) - px(x+1)\},$$

where $x = [2k/p]$, the inequality in the theorem follows. By Lemma 6.2.2, equality holds if and only if $n_{dij} = x$ or $x+1$. □

Note that if $2k < p$ then $x = 0$ and in such a case,

$$\operatorname{tr}(C_d) \le 2b(k-1). \tag{6.6.2}$$

Suppose now that there is a design $d^* \in \mathcal{D}(p, b, k)$ such that (i) C_{d^*} is completely symmetric and, (ii) $\operatorname{tr}(C_{d^*})$ attains the upper bound given by (6.6.1), then by Theorem 6.2.1, d^* is universally optimal for g.c.a. effects in $\mathcal{D}(p, b, k)$.

In Section 5.5, we have seen how a nested balanced incomplete block design with sub-block size two can be converted into an incomplete block design for diallel crosses. Suppose d is a nested BIB design with parameters $v = p, b_1, b_2, k_1, k_2 = 2, r$, following the notations of Definition 3.7.1. Then a design d^* for diallel crosses involving $p(p-1)/2$ crosses and $b = b_1$ blocks each of size $k = k_1/2$ can be obtained as indicated in Section 5.5. Each cross is replicated $2b_2/\{p(p-1)\}$ blocks in d^*. Then, $d^* \in \mathcal{D}(p, b, k)$. Also, from (5.5.15), the coefficient matrix of the reduced normal equations for g.c.a. effects using d^* is

$$C_{d^*} = 2(p-1)^{-1}b(k-1)(I_p - p^{-1}J_p). \tag{6.6.3}$$

Clearly, C_{d^*} is completely symmetric, $2k = k_1 < v = p$ and $\mathrm{tr}(C_{d^*}) = 2b(k-1)$, which equals the upper bound given by (6.6.2). Thus, it follows that the design d^* is universally optimal for g.c.a. effects in $\mathcal{D}(p, b, k)$. Under d^*, the variance of the best linear unbiased estimator of any elementary contrast among the g.c.a. effects is

$$\frac{(p-1)\sigma^2}{b(k-1)}, \tag{6.6.4}$$

where σ^2 is the per observation variance.

Next, recall the connection between a two-associate triangular PBIB design, say d_1 and an incomplete block design for diallel crosses say d^*, as described in Section 5.5. Suppose the parameters of the triangular design d_1 are $v = p(p-1)/2, b, r, k, \lambda_1, \lambda_2$, where p is the number of inbred lines in the diallel cross experiment. For the design d^*, we have from (5.5.16),

$$C_{d^*} = \theta(I_p - p^{-1}J_p), \tag{6.6.5}$$

with θ as given by (5.5.17). Hence,

$$\mathrm{tr}(C_{d^*}) = k^{-1}p(p-1)\{r(k-1) - (p-2)\lambda_1\}. \tag{6.6.6}$$

Also, from Theorem 6.6.1, for an arbitrary design $d \in \mathcal{D}(p, b, k)$, $\mathrm{tr}(C_d)$ is bounded above by

$$k^{-1}b\{2k(k-1-2x) + px(x+1)\}, \tag{6.6.7}$$

where $x = [2k/p]$. Equating (6.6.6) and (6.6.7), we have the following result.

Theorem 6.6.2 *An incomplete block design for diallel crosses derived from a triangular design with parameters $v = p(p-1)/2, b, r, k, \lambda_1, \lambda_2$ is universally optimal over $\mathcal{D}(p, b, k)$ if*

$$p(p-1)(p-2)\lambda_1 = bx\{4k - p(x+1)\} \tag{6.6.8}$$

where $x = [2k/p]$.

Now, for a triangular design with parameters $v = p(p-1)/2, b, r, k, \lambda_1 = 0, \lambda_2$, it can be seen that $2k \leq p$ (Exercise 4.15). It follows then that with $\lambda_1 = 0$, the condition (6.6.8) always holds. This leads to the following corollary to Theorem 6.6.2, obtained earlier by Dey and Midha (1996).

Corollary 6.6.1 *A diallel cross design obtained via a triangular design with $\lambda_1 = 0$ is universally optimal over $\mathcal{D}(p, b, k)$.*

Triangular designs not satisfying the condition $\lambda_1 = 0$ can also lead to universally optimal designs for diallel crosses; for details, see Das *et al.* (1998).

So far in this section, we have restricted the search for an optimal incomplete block design for diallel crosses under a model that does not include the specific combining ability (s.c.a.) effects. Suppose the model is modified to include the s.c.a. effects as well, apart from the g.c.a. effects and the block effects. The interest of the experimenter may still be in optimally estimating contrasts among the g.c.a. effects, but in the presence of s.c.a. effects. Chai and Mukerjee (1999) have shown that the diallel cross designs derived from triangular designs satisfying (6.6.8) remain optimal for the g.c.a. effects even when the s.c.a. effects are included in the model. Thus, the findings in Dey and Midha (1996) and Das *et al.* (1998) on the optimality of diallel cross designs derived from triangular designs satisfying (6.6.8) remain robust under a model that includes s.c.a. effects as well.

Turning to the issue of s.c.a. effects themselves, Chai and Mukerjee (1999) proved the following result.

Theorem 6.6.3 *Let d be a triangular PBIB design with parameters $v = p(p-1)/2, b, r, k, \lambda_1 > 0, \lambda_2 = 0$ and d_0 be a design derived out of it for a diallel cross experiment. Then, d_0 is universally optimal in $\mathcal{D}(p, b, k)$ for any complete set of orthonormal contrasts representing the s.c.a. effects.*

With a large number of lines, p, a complete diallel cross experiment involving $p(p-1)/2$ crosses may become prohibitively large. In such a situation, it might often be necessary to experiment only with a subset of all the possible $\binom{p}{2}$ crosses. Such a subset of crosses is referred to as a *partial* diallel cross. Several partial diallel cross plans are available in the literature; see e.g., Arya (1983), Curnow (1963), Hinkelmann and Kempthorne (1963) and Singh and Hinkelmann (1995). However, even in the unblocked situation, the issue of finding an *optimal* partial diallel cross plan has not received much attention. Having chosen an optimal diallel cross plan, further blocking of the crosses might be necessary to control the error. We briefly discuss below some developments in this area.

We consider the model (5.5.12), where, to begin with we consider an unblocked situation, i.e., the block effects are absent from (5.5.12).

Suppose $p = mn$ where $m \geq 2, n \geq 3$ are integers. Let us partition the set $\{1, 2, \ldots, p\}$ into m mutually exclusive and exhaustive subsets $S_1, \ldots, S_m$, each having n elements. Let

$$d^* = \{(i \times j) : 1 \leq i < j \leq p \text{ and } i, j \in S_u \text{ for some } u\}. \tag{6.6.9}$$

If $\mathcal{D}(N, p)$ denotes the class of all N-observation partial diallel cross plans with $N < \binom{p}{2}$, then clearly, $d^* \in \mathcal{D}(N, p)$, where $N = mn(n-1)/2$. Mukerjee (1997) proved the following result.

Theorem 6.6.4 *For each $m \geq 2$ and $n \geq 3$, the plan d^* is uniquely (up to isomorphism) E-optimal in $\mathcal{D}(N, p)$ where $N = mn(n-1)/2$. Furthermore, the plan d^* is uniquely D- and A-optimal in $\mathcal{D}(N, p)$ for $n = 3$.*

Though the D- and A-optimality of d^* for $n \geq 4$ has not yet been established, Mukerjee (1997) observed through numerical investigations that d^* has very high efficiency under the D- and A-criteria for many practical values of m and $n \geq 4$. Turning to the question of finding optimal incomplete block designs for partial diallel crosses, Mukerjee (1997) obtained two classes of E-optimal incomplete block designs for the following two cases: $n \geq 5$, n odd, and $n \geq 4$, n even. We refer to the original source for details.

For some other related results, see Das, Dean and Gupta (1998), Gupta, Das and Kageyama (1994) and Das and Dey (2004). In conclusion, we remark that the problem of finding optimal incomplete block designs for partial diallel cross experiments is still wide open.

6.7 Exercises

6.1. Provide a proof of Lemma 6.2.1.

6.2. Show that a two-associate PBIB design obtained by treating the points of a partial geometry (r, k, t) as treatments and the lines as blocks is ψ_f-optimal over the class of equireplicate, binary designs for any convex function f such that f' is strictly concave and $f(0) = \lim_{x \to 0^+} = \infty$.

6.3. Show that the variance of the BLUE of any elementary treatment contrast using a design $d \in \mathcal{D}_0(v, b, k)$ is an even multiple of σ^2, where σ^2 is the variance of an observation and the class $\mathcal{D}_0$ is as in Section 6.3.3.

6.4. Establish the (M, S)-optimality and type I optimality of the designs d^* defined in Remark 6.3.3.

6.5. Provide a proof of Lemma 6.3.3.

6.6. Prove the E- and A-optimality of the design d^* in Remark 6.3.5.

6.7. With C_d, M_d as in (6.4.1) and $L = (-\mathbf{1}_v, I_v)$, show that $M_d = (LC_d^- L')^{-1}$.

6.8. Provide a proof of Theorem 6.4.2.

6.9. Suppose d_1 is an L-design with parameters $v = 2m$ treatments (doses) and b blocks of size $k < v$ each. Using the notations of Section 6.5, show that $\operatorname{tr}(\mathbb{D}(G\hat{\tau})_{d_1}) = 3v\sigma^2/(bk)$.

6.10. Prove Corollary 6.6.1.

Appendix

The Appendix has four sections. In Section A.1, some useful results in linear algebra are summarized. Some basic results in linear statistical models are given in Section A.2. Several construction methods discussed in this book use results in finite (or, Galois) fields. In Section A.3, we describe the essentials of Galois fields. In Section A.4, some essential concepts and results on finite projective and Euclidean geometries are described.

A.1 Some Results in Linear Algebra

In this section, we summarize some notations, terminology and basic results in linear algebra which are used in the earlier chapters. We deal exclusively with *real* matrices and vectors. All vectors are written as column vectors, and such vectors are denoted by boldface numerals or letters. A prime over a matrix or vector denotes its transpose. For a positive integer s, $\mathbf{1}_s$ and I_s respectively, denote an $s \times 1$ vector of all ones and an identity matrix of order s. For positive integers a, b, $\mathbf{0}_{ab}$ denotes an $a \times b$ null matrix and J_{ab}, an $a \times b$ matrix of all ones; J_{aa} is simply denoted by J_a. Similarly, $\mathbf{0}_{a1}$ is denoted by $\mathbf{0}_a$. The subscripts are suppressed when there is no confusion regarding the order of the matrices involved. A diagonal matrix with diagonal entries $a_1, a_2, \ldots, a_n$ is written as $\text{diag}(a_1, a_2, \ldots, a_n)$.

A.1.1. Let A be a square matrix. Then, A is called symmetric if $A' = A$. The trace of A (i.e., the sum of the entries on the principal diagonal) and the determinant of A are denoted by $\text{tr}(A)$ and $\det(A)$, respectively.

A.1.2. Let A be an $m \times n$ matrix. The vector space spanned by the columns of A, called the column space or range space of A, is denoted by $\mathcal{C}(A)$. Similarly, the vector space spanned by the rows of A, called the row space of A, is denoted by $\mathcal{R}(A)$. The dimensions of $\mathcal{C}(A)$ and $\mathcal{R}(A)$ are equal, and their common value is called the rank of A and denoted by $Rank(A)$. The collection of vectors $\{x\}$ satisfying $Ax = \mathbf{0}$ forms a vector space, called the null space of A, which is denoted by $\mathcal{N}(A)$. The dimension of the null space of a matrix A is called the nullity of A. The rank and nullity of a matrix A add up to the number of columns in A.

A.1.3. A square matrix A of order a is called *completely symmetric* if $A = \alpha I_a + \beta J_a$ for some scalars α and β. A square matrix A is said to be *idempotent* if $A^2 = A$ and, for an idempotent matrix A, $Rank(A) = \mathrm{tr}(A)$. A square matrix A of order n is called *orthogonal* if $AA' = I_n$.

A.1.4. Suppose A is an $m \times n$ matrix and let $B = AA'$. Then,

$$\mathcal{C}(B) = \mathcal{C}(A).$$

A.1.5. For a matrix A, A^- denotes an arbitrary generalized inverse (g-inverse) of A, i.e., A^- is a solution of the matrix equation $AXA = A$. Note that A^- is non-unique, unless A is square and invertible. Also, $Rank(A) \leq Rank(A^-)$. For a matrix A, we let A^+ to denote the (unique) Moore-Penrose inverse of A, i.e., A^+ satisfies the following conditions:

(i) $AA^+A = A$, (ii) $A^+AA^+ = A^+$, (iii) AA^+ is symmetric, and
(iv) A^+A is symmetric.
It may be noted that for any matrix A, $Rank(A^+) = Rank(A)$.

A.1.6. Let A and B be non-null matrices. Then the product AC^-B is invariant with respect to the choice of the g-inverse if and only if $\mathcal{C}(B) \subset \mathcal{C}(C)$ and $\mathcal{R}(A) \subset \mathcal{R}(C)$.

A.1.7. Let A be an $m \times n$ matrix and suppose B is an arbitrary g-inverse of $A'A$, i.e., $B = (A'A)^-$. Then, the following are true:
(a) B' is also a g-inverse of $A'A$.
(b) $ABA'A = A$, i.e., BA' is a g-inverse of A.
(c) ABA' is invariant with respect to the choice of B.
(d) ABA' is symmetric, whether or not B is symmetric. Also, ABA' is idempotent.

A.1.8. For a pair of matrices E, F, $E \otimes F$ denotes the Kronecker (tensor) product of E and F, i.e., if $E = (e_{ij})$, then $E \otimes F = (e_{ij}F)$. The following are some well known and easily verifiable facts about tensor product of matrices:

(i) $(A \otimes B)(C \otimes D) = AC \otimes BD$, provided the products AC and BD are well defined.
(ii) $(A_1 + A_2) \otimes B = A_1 \otimes B + A_2 \otimes B$.
(iii) $A \otimes (B_1 + B_2) = A \otimes B_1 + A \otimes B_2$.
(iv) For scalars a, b, $aA \otimes bB = ab(A \otimes B)$.
(v) $(A \otimes B)' = A' \otimes B'$.
(vi) If A, B are invertible matrices, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

A.1.9. Let A be a square matrix of order n. The determinantal equation $\det(A - \lambda I_n) = 0$ is called the characteristic equation of A. The roots of this equation are called the eigenvalues (or, latent roots or characteristic roots) of A. If λ_i is an eigenvalue of a square matrix A, then there exist non-null vectors $\boldsymbol{x}_i$ and $\boldsymbol{y}_i$ such that

$$A\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i \quad \text{and} \quad \boldsymbol{y}_i' A = \lambda_i \boldsymbol{y}_i'.$$

The vectors $\boldsymbol{x}_i$ and $\boldsymbol{y}_i$ are called the right and left eigenvectors, respectively, of A corresponding to the eigenvalue λ_i. Clearly, if A is symmetric, the left and right eigenvectors corresponding to an eigenvalue are the same. The following are some useful and well-known facts regarding the eigenvalues and eigenvectors of a (real) symmetric matrix A of order n:

(i) All the eigenvalues of A are real and the eigenvectors can be chosen to be real.
(ii) The eigenvectors corresponding to distinct eigenvalues of A are orthogonal. Also, if A has an eigenvalue λ_i with multiplicity m_i, then there exist m_i mutually orthogonal eigenvectors of A corresponding to λ_i. Throughout, the multiplicity of an eigenvalue means its algebraic multiplicity. An eigenvalue with multiplicity 1 is called a simple eigenvalue.
(iii) If $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A, including multiplicities, and $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n$ be the corresponding mutually orthogonal eigenvectors each chosen to be of unit length, then

$$A = \lambda_1 \boldsymbol{\xi}_1 \boldsymbol{\xi}_1' + \cdots + \lambda_n \boldsymbol{\xi}_n \boldsymbol{\xi}_n'.$$

Such a representation of a symmetric matrix A is called its spectral representation. Equivalently, the above can be written as

$$A = P \Delta P',$$

where the matrix P, with columns $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_n$ is orthogonal, and $\Delta = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$.

Let A be a symmetric matrix and G be a g-inverse of A such that GA is also symmetric. Then, the reciprocal of a non-zero eigenvalue of A is an eigenvalue of G. In particular, for a symmetric matrix A, the non-zero eigenvalues of A^+ (the Moore-Penrose inverse of A) are the reciprocals of the non-zero eigenvalues of A.

A.1.10. For *any* matrix A (not necessarily square), the symmetric matrices AA' and $A'A$ have the same set of positive eigenvalues with the same multiplicities.

A.1.11. Let A be a symmetric matrix of order n and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of A. Then

$$\max_{\boldsymbol{u} \neq \mathbf{0}} \frac{\boldsymbol{u}'A\boldsymbol{u}}{\boldsymbol{u}'\boldsymbol{u}} = \lambda_1, \quad \min_{\boldsymbol{u} \neq \mathbf{0}} \frac{\boldsymbol{u}'A\boldsymbol{u}}{\boldsymbol{u}'\boldsymbol{u}} = \lambda_n.$$

A.1.12. For any matrix A, we let $\mathrm{pr}(A)$ to denote the orthogonal projection matrix onto $\mathcal{C}(A)$, the column space of A. Then, $\mathrm{pr}(A)$ is given by $\mathrm{pr}(A) = A(A'A)^- A'$. The orthogonal projection matrix onto the space that is orthogonal to $\mathcal{C}(A)$ is denoted by $\mathrm{pr}^{\perp}(A) = I - \mathrm{pr}(A)$, where I is an identity matrix of appropriate order. From A.1.7, it follows that $\mathrm{pr}(A)$ (and consequently, $\mathrm{pr}^{\perp}(A)$) is a symmetric, idempotent matrix.

For a partitioned matrix $A = [B \;\; C]$, it can be seen that

$$\mathrm{pr}^{\perp}(A) = \mathrm{pr}^{\perp}(B) - \mathrm{pr}^{\perp}(B)C(C'\mathrm{pr}^{\perp}(B)C)^- C'\mathrm{pr}^{\perp}(B).$$

A.2 Some Aspects of Linear Models

In this section, we briefly describe some essential results in linear (statistical) models. For proofs of the results below and other details, a reference may be made e.g., to Chapter 4 of Rao (1973). Let $Y_1, \ldots, Y_n$ be uncorrelated observations such that for $1 \leq i \leq n$,

$$\begin{aligned} \mathbb{E}(Y_i) &= x_{i1}\beta_1 + x_{i2}\beta_2 + \cdots + x_{ip}\beta_p, \\ \mathrm{Var}(Y_i) &= \sigma^2, \end{aligned} \tag{A.2.1}$$

where $\beta_1, \ldots, \beta_p$ and σ^2 are unknown parameters and $\{x_{ij}\}$ are known elements. A specification of the type (A.2.1) is called a linear model. Setting $\boldsymbol{Y} = (Y_1, \ldots, Y_n)'$, $\boldsymbol{\beta} = (\beta_1 \ldots, \beta_p)'$ and $X = (x_{ij})$, one can write (A.2.1) in matrix notation as

$$\boldsymbol{Y} = X\boldsymbol{\beta} + \boldsymbol{\epsilon}, \; \mathbb{E}(\boldsymbol{\epsilon}) = \mathbf{0}, \; \mathbb{D}(\boldsymbol{\epsilon}) = \sigma^2 I_n, \tag{A.2.2}$$

where $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)'$ and for $1 \le i \le n$, ϵ_i is the random error term associated with the observation Y_i. These error terms are uncorrelated random variables with zero expectations and finite variance σ^2. In the above, $\mathbb{E}(\cdot)$ and $\mathbb{D}(\cdot)$ as before are respectively, the expectation and dispersion operators. From (A.2.2), it follows that

$$\mathbb{E}(\boldsymbol{Y}) = X\boldsymbol{\beta},\ \mathbb{D}(\boldsymbol{Y}) = \sigma^2 I_n. \tag{A.2.3}$$

The problem is to estimate the unknown parameters in the model (A.2.1) (equivalently, (A.2.3)) and draw inferences on them. We first state the following well-known facts:
(i) If $\boldsymbol{l}'\boldsymbol{Y}, \boldsymbol{m}'\boldsymbol{Y}, \ldots$ denote linear functions of observations, then

$$\mathbb{E}(\boldsymbol{l}'\boldsymbol{Y}) = \boldsymbol{l}'X\boldsymbol{\beta},\ \mathrm{Var}(\boldsymbol{l}'\boldsymbol{Y}) = \sigma^2\boldsymbol{l}'\boldsymbol{l},\ \mathrm{Cov}(\boldsymbol{l}'\boldsymbol{Y}, \boldsymbol{m}'\boldsymbol{Y}) = \sigma^2\boldsymbol{l}'\boldsymbol{m}.$$

(ii) The following result is also useful.

Lemma A.2.1 *If $\boldsymbol{x}$ is a random vector with $\mathbb{E}(\boldsymbol{x}) = \boldsymbol{\mu}$, $\mathbb{D}(\boldsymbol{x}) = V$ and A is a symmetric matrix, then*

$$\mathbb{E}(\boldsymbol{x}'A\boldsymbol{x}) = \mathrm{tr}(AV) + \boldsymbol{\mu}'A\boldsymbol{\mu}.$$

We apply the method of least squares to estimate the parameter $\boldsymbol{\beta}$ of the model (A.2.2), which involves the minimization of the error sum of squares S, given by

$$S = \boldsymbol{\epsilon}'\boldsymbol{\epsilon} = (\boldsymbol{Y} - X\boldsymbol{\beta})'(\boldsymbol{Y} - X\boldsymbol{\beta}) = \sum_{i=1}^{n}(Y_i - x_{i1}\beta_1 - \cdots - x_{ip}\beta_p)^2. \tag{A.2.4}$$

Differentiating S with respect to $\boldsymbol{\beta}$ and equating the derivative to zero, one obtains the following set of linear equations, called the *normal equations*:

$$X'X\boldsymbol{\beta} = X'\boldsymbol{Y}. \tag{A.2.5}$$

It is not hard to see that the normal equations (A.2.5) are consistent (i.e., admit a solution), whatever be the rank of X. If $\hat{\boldsymbol{\beta}}$ is a solution of (A.2.5), then it can be shown that the minimum of

$$S = (\boldsymbol{Y} - X\boldsymbol{\beta})'(\boldsymbol{Y} - X\boldsymbol{\beta})$$

is attained at $\boldsymbol{\beta} = \hat{\boldsymbol{\beta}}$ and is unique for all solutions $\hat{\boldsymbol{\beta}}$ of (A.2.5).

Let $\boldsymbol{c}'\boldsymbol{\beta} = c_1\beta_1 + \cdots + c_p\beta_p$ be a linear parametric function. The least squares estimator of $\boldsymbol{c}'\boldsymbol{\beta}$ is defined to be $\boldsymbol{c}'\hat{\boldsymbol{\beta}}$ where $\hat{\boldsymbol{\beta}}$ is any solution of

(A.2.5). A linear parametric function $c'\beta$ is said to be *estimable* under the model (A.2.3) if there exists a linear function of observations, say $l'Y$, such that $\mathbb{E}(l'Y) = c'\beta$ for all $\beta \in \mathbb{R}^p$.

A linear parametric function $c'\beta$ is estimable if and only if $c \in \mathcal{C}(X')$ or equivalently, if and only if $c \in \mathcal{C}(X'X)$.

If $c'\beta$ is an estimable function, then its least squares estimator $c'\hat{\beta}$

(a) is linear in Y, unbiased for $c'\beta$, and unique for all solutions $\hat{\beta}$ of (A.2.5),

(b) has the smallest variance in the class of linear unbiased estimators of $c'\beta$.

In view of (a) and (b) above, $c'\hat{\beta}$ is called the best linear unbiased estimator (BLUE) of $c'\beta$.

Let $p'\beta, q'\beta$ be a pair of estimable functions and $p'\hat{\beta}, q'\hat{\beta}$ be their respective least squares estimators. Also, let G denote an arbitrary g-inverse of $X'X$. Then,

$$\mathrm{Var}(p'\hat{\beta}) = \sigma^2 p'Gp, \ \mathrm{Cov}(p'\hat{\beta}, q'\hat{\beta}) = \sigma^2 p'Gq.$$

A slightly more general model than (A.2.3) was considered by Aitken (1935), which incorporates correlations between the error terms and is given by

$$\mathbb{E}(Y) = X\beta, \ \ \mathbb{D}(Y) = \sigma^2 U, \tag{A.2.6}$$

where U is a known positive definite matrix. The model (A.2.6) can be reduced to (A.2.3) by making the transformation $Z = U^{-\frac{1}{2}}Y$, where $U^{-\frac{1}{2}}$ is the inverse of $U^{\frac{1}{2}}$, the unique square root of U. Under this transformation, we have

$$\mathbb{E}(Z) = U^{-\frac{1}{2}}X\beta = V\beta, \ \ \mathbb{D}(Z) = \sigma^2 I, \tag{A.2.7}$$

where $V = U^{-\frac{1}{2}}X$. A particular case of (A.2.6) is

$$\mathbb{E}(Y) = X\beta, \ \ \mathbb{D}(Y) = \Sigma, \tag{A.2.8}$$

where Σ is a known positive definite matrix. Using the transformation $Z = \Sigma^{-\frac{1}{2}}Y$, one can again reduce (A.2.8) to the model (A.2.3). It follows then that the normal equations under the model (A.2.6) are

$$X'G^{-1}X\beta = X'G^{-1}Y, \tag{A.2.9}$$

and those under the model (A.2.8) are

$$X'\Sigma^{-1}X\beta = X'\Sigma^{-1}Y. \tag{A.2.10}$$

Next, consider the problem of estimating σ^2. Let R_0^2 denote the minimum error sum of squares, i.e.,

$$R_0^2 = \min_{\boldsymbol{\beta}}(\boldsymbol{Y} - X\boldsymbol{\beta})'(\boldsymbol{Y} - X\boldsymbol{\beta}) = (\boldsymbol{Y} - X\hat{\boldsymbol{\beta}})'(\boldsymbol{Y} - X\hat{\boldsymbol{\beta}}),$$

where $\hat{\boldsymbol{\beta}}$, as before, is a solution of (A.2.5). Then, one can verify the following alternative expressions for R_0^2:

$$\begin{aligned} R_0^2 &= \boldsymbol{Y}'\boldsymbol{Y} - \boldsymbol{Y}'X\hat{\boldsymbol{\beta}} = \boldsymbol{Y}'\boldsymbol{Y} - \hat{\boldsymbol{\beta}}'X'X\hat{\boldsymbol{\beta}} \\ &= \boldsymbol{Y}'\boldsymbol{Y} - \boldsymbol{Y}'X(X'X)^{-}X'\boldsymbol{Y} \\ &= \boldsymbol{Y}'(I - X(X'X)^{-}X')\boldsymbol{Y} \\ &= \boldsymbol{Y}'\mathrm{pr}^{\perp}(X)\boldsymbol{Y}. \end{aligned} \tag{A.2.11}$$

The vector $\boldsymbol{e} = (\boldsymbol{Y} - X\hat{\boldsymbol{\beta}})$ is called the residual vector. The following are some results on the residual vector:

(i) $\mathbb{E}(\boldsymbol{e}) = \boldsymbol{0}$.

(ii) $\mathbb{D}(\boldsymbol{e}) = \mathbb{D}(\boldsymbol{Y}) - \mathbb{D}(X\hat{\boldsymbol{\beta}}) = \sigma^2(I - X(X'X)^{-}X')$.

(iii) Let $\boldsymbol{l}'\boldsymbol{Y}$ be a linear function of observations such that $\mathbb{E}(\boldsymbol{l}'\boldsymbol{Y}) = 0$ (such linear functions are called zero functions or, *error functions*). Then, $\mathrm{Cov}(\boldsymbol{p}'\hat{\boldsymbol{\beta}}, \boldsymbol{l}'\boldsymbol{Y}) = 0$, where $\boldsymbol{p}'\hat{\boldsymbol{\beta}}$ is the least squares estimator of an estimable function $\boldsymbol{p}'\boldsymbol{\beta}$. In particular, by virtue of (i) above,

$$\mathrm{Cov}(\boldsymbol{p}'\hat{\boldsymbol{\beta}}, \boldsymbol{e}) = 0.$$

(iv) $\mathbb{E}(R_0^2) = \mathbb{E}(\boldsymbol{e}'\boldsymbol{e}) = (n-r)\sigma^2$, where $r = Rank(X)$. Thus, an unbiased estimator of σ^2 is

$$\hat{\sigma^2} = R_0^2/(n-r).$$

Finally, we consider the problem of testing of hypothesis involving estimable linear parametric functions. To that end, we make the additional assumption that the observation vector follows an n-variate normal distribution, i.e., $\boldsymbol{Y} \sim N_n(X\boldsymbol{\beta}, \sigma^2 I_n)$. Let $\boldsymbol{p}_1'\boldsymbol{\beta}, \ldots, \boldsymbol{p}_k'\boldsymbol{\beta}$ be a set of k independent parametric functions, each of which is estimable under the model (A.2.3). We can write the above as $P'\boldsymbol{\beta}$, where P' is a $k \times p$ matrix whose ith row is $\boldsymbol{p}_i', 1 \le i \le k$ and P' has full row rank. Let it be required to test the hypothesis

$$H_0 : P'\boldsymbol{\beta} = \boldsymbol{\theta}_0,$$

where $\boldsymbol{\theta}_0$ is a known $k \times 1$ vector. Define R_1^2 to be the minimum of $(\boldsymbol{Y} - X\boldsymbol{\beta})'(\boldsymbol{Y} - X\boldsymbol{\beta})$, subject to the hypothesis H_0, i.e.,

$$R_1^2 = \min_{\boldsymbol{\beta}: P'\boldsymbol{\beta} = \boldsymbol{\theta}_0} (\boldsymbol{Y} - X\boldsymbol{\beta})'(\boldsymbol{Y} - X\boldsymbol{\beta}). \tag{A.2.12}$$

Then, a test for the hypothesis H_0 is provided by the statistic

$$\mathcal{F} = \frac{(R_1^2 - R_0^2)/k}{R_0^2/(n-r)}, \tag{A.2.13}$$

where R_0^2 is as in (A.2.11). The statistic $\mathcal{F}$ under H_0 has a central F-distribution on k and $n-r$ degrees of freedom. Thus, if $F_{\alpha;k,n-r}$ is the upper α percent point of an F-distribution with k and $n-r$ degrees of freedom, then one rejects H_0 at α percent level of significance if $\mathcal{F} > F_{\alpha;k,n-r}$.

A.3 Finite Fields

In this section, we briefly discuss the essential notions in finite fields. More details can be found in the books by McCoy (1948), Jacobson (1964) or, Lidl and Niederreiter (1986).

Suppose S is a nonempty set. A *binary operation* on S is a map from $S \times S$ to S. For instance, if S is the set of all integers, then addition, subtraction and multiplication are examples of binary operations; however, division on the set of integers is not a binary operation.

Suppose F is a set and $+$ and $\cdot$ are two binary operations over F. We shall call the operations $+$ and $\cdot$ as "addition" and "multiplication", respectively, though these can be different from the usual operations of addition and multiplication of real numbers. The system $\mathcal{F} = (F, +, \cdot)$ is a *field* if the following axioms hold:

F1: For all $a, b \in F$, $a + b = b + a$.

F2: For all $a, b, c \in F$, $(a + b) + c = a + (b + c)$.

F3: There exists a unique element $0 \in F$ such that $a + 0 = 0 + a = a$ for all $a \in F$. The element $0 \in F$ is the additive identity of the field.

F4: For each element $a \in F$, there exists a unique element $-a \in F$ such that $a + (-a) = 0$. The element $-a$ is called the additive inverse of a.

F5: $a \cdot b = b \cdot a$ for all $a, b \in F$.

F6: For all $a, b, c \in F$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

F7: There exists a unique element $1 \in F$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in F$. The element 1 is called the multiplicative identity of the field.

F8: Let $F_0 = F \setminus \{0\}$. For each $a \in F_0$, there exists a unique element a^{-1}, called the multiplicative inverse of a, such that $a \cdot a^{-1} = 1$.

F9: For all $a, b, c \in F$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

Clearly for example, the set of real numbers with the usual operations of addition and multiplication of reals forms a field. We shall sometimes denote the field $\mathcal{F}$ and set associated with it by the same symbol F. If the set F is finite, the corresponding field is called a finite field or, a *Galois* field, called after the French mathematician Évariste Galois (1811-1832). In such a case, the number of elements in F is called the *order* of the field. In this book, we are primarily concerned with Galois fields. To describe the essentials of a Galois field, we need the notion of congruence. The quantity a is said to be *congruent* to b modulo m if m divides $a - b$ and we write this as $a \equiv b \pmod{m}$. If $x \equiv a \pmod{m}$, then a is called the *residue* of x mod m.

Suppose p is a prime number. Let F_p be the set of residues modulo p, i.e., $F_p = \{0, 1, \ldots, p-1\}$. Then $\mathcal{F}_p = (F_p, +_p, \cdot_p)$ forms a field of p elements where $+_p$ and $\cdot_p$ are addition and multiplication modulo p. However, this is not true if p is not a prime number. The field $\mathcal{F}_p$ where p is a prime will often be denoted by $GF(p)$. There are Galois fields other than $GF(p)$. It can be shown that the order of a Galois field must be a power of a prime. How does one construct a Galois field of order $s = p^n$ where p is a prime and $n > 1$ is an integer? Towards that goal, we first need the notion of a ring. A *ring* is a system that satisfies all the axioms of a field *except* F5, F7 and F8. If axioms F1–F7 and F9 hold, then we have a *commutative ring with unity.*

Let $F[x]$ be the set of all polynomials in the variable x with coefficients from a field F, that is

$$F[x] = \{a_0 + a_1 x + \cdots + a_k x^k : k \text{ a nonnegative integer, } a_0, \ldots, a_k \in F\}.$$

The zero polynomial is one for which all coefficients are zero. The degree of a nonzero polynomial $f(x) \in F[x]$, denoted by $deg(f)$ is the number k if $f(x) = a_0 + a_1 x + \cdots a_k x^k$ with $a_k \neq 0$.

The binary operations $+$ and $\cdot$ on F induce binary operations on the members of $F[x]$ with the usual rules of addition and multiplication of polynomials. For example, consider the Galois field $GF(3)$ which has elements 0,1 and 2. If we take the polynomials

$$\begin{aligned} f_1(x) &= 1 + x^3 - x^4 \\ f_2(x) &= x + x^2 - x^4 + x^5, \end{aligned}$$

then

$$\begin{aligned} f_1(x) + f_2(x) &= 1 + x + x^2 + x^3 + x^4 + x^5, \\ f_1(x) \cdot f_2(x) &= x + x^2 + x^5 - x^6 - x^7 - x^8 - x^9. \end{aligned}$$

The triple $(F[x], +, \cdot)$ however is not a field but a commutative ring with unity. This is referred to as a polynomial ring induced by the field F. Suppose $f(x), g(x) \in F[x]$ where F is a field. If $g(x)$ is a nonzero polynomial, then there exist unique polynomials $r(x)$ and $s(x)$, both belonging to $F[x]$, such that

$$f(x) \equiv r(x)g(x) + s(x) \text{ and } deg(s(x)) < deg(g(x)).$$

The polynomial $s(x)$ is called the residue of $f(x)$ with respect to $g(x)$, written as

$$f(x) \equiv s(x) \pmod{g(x)}.$$

For any fixed nonzero polynomial $g(x) \in F[x]$, one can now define an equivalence relation $\sim$ on $F[x]$ as follows: if $f_1(x), f_2(x) \in F[x]$ then $f_1(x) \sim f_2(x)$ if $f_1(x) - f_2(x) \equiv 0 \pmod{g(x)}$. If $deg(g(x)) = n$, this equivalence relation has p^n equivalence classes where p is the order of the field F. Representatives of these p^n classes are

$$a_0 + a_1 x + \cdots + a_{n-1} x^{n-1},\ a_i \in F,\ 0 \le i \le n-1.$$

The set of these equivalence classes is denoted by $F[x]/(g(x))$.

A nonzero polynomial $f(x) \in F[x]$ is said to be *irreducible* over F if for any $f_1(x), f_2(x) \in F[x]$ with $f(x) = f_1(x)f_2(x)$, either $deg(f_1(x)) = 0$ or $deg(f_2(x)) = 0$. Otherwise $f(x)$ is called reducible. For example, consider $GF(3)$. Then the polynomial $f(x) = x^2 + x + 2$ is irreducible over $GF(3)$. As another example, consider $GF(5)$. Then, the polynomial $x^2 + 1$ is reducible over $GF(5)$ as,

$$x^2 + 1 = (x + 2)(x + 3).$$

However, the polynomial $x^2 + 2$ can be checked to be irreducible over $GF(5)$.

We are now in a position to state the following result.

Theorem A.3.1 *Let F be a Galois field and let $g(x) \in F[x]$, $g(x) \neq 0$. The system $(F[x]/(g(x)), +, \cdot)$ is a Galois field if and only if $g(x)$ is irreducible over F.*

Thus a Galois field of order $s = p^n$ where p is a prime and n, a positive integer can be constructed as follows: (i) Take F to be the field of residues modulo p with addition and multiplication modulo p as the binary operations; (ii) select an irreducible polynomial $g(x) \in F[x]$ of degree n. Then $F[x]/(g(x))$ provides a Galois field.

For example, let us construct a Galois field of order 2^3. The polynomial $f(x) = x^3 + x + 1$ is seen to be irreducible over $GF(2)$. Thus the desired Galois field is given by $F[x]/(f(x))$. The elements of this field can be represented by $0, 1, x, x+1, x^2, x^2+1, x^2+x, x^2+x+1$. It is interesting to observe that the successive powers of x in this field generate all the nonzero elements of the field as, $x^1 = x, x^2 = x^2, x^3 = x+1, x^4 = x^2+x, x^5 = x^2+x+1, x^6 = x^2+1, x^7 = 1$. An element of a field with this property is called a *primitive element* of the field. An irreducible polynomial $f(x)$ over $GF(p)$ of degree n is said to a be a *primitive polynomial* if x is a primitive element in the field $GF(p^n) = F[x]/(f(x))$. Some of the primitive polynomials are listed below for ready reference.

$GF(p^n)$	Primitive Polynomial over $GF(p)$
2^2	$x^2 + x + 1$
2^3	$x^3 + x + 1$
2^4	$x^4 + x + 1$
2^5	$x^5 + x^2 + 1$
3^2	$x^2 + x + 2$
3^3	$x^3 + 2x + 1$
5^2	$x^2 + x + 2$
7^2	$x^2 + x + 3$

The following table lists a primitive element in a field $GF(p)$ for some values of the prime p.

p	Primitive Element	p	Primitive Element
3	2	17	3
5	2	19	2
7	3	23	5
11	2	29	2
13	2	31	3

A.4 Finite Geometries

As seen earlier in this book, finite projective and Euclidean geometries have close connections with incomplete block designs. In this section, we briefly describe some aspects of finite geometries. For comprehensive descriptions on finite geometries, we refer to Dembowski (1968) and Hirschfeld (1979).

Let p be a prime number and q, a positive integer and let $s = p^q$. An ordered set of $n+1$ elements $(x_0, x_1, \ldots, x_n)$, where x_i's belong to $GF(s)$ and $(x_0, x_1, \ldots, x_n) \neq (0, , \ldots, 0)$, is called a point in the finite projective geometry, $PG(n, s)$. Two ordered sets $(x_0, x_1, \ldots, x_n)$ and $(y_0, y_1, \ldots, y_n)$ represent the same point in $PG(n, s)$ if and only if $y_i = cx_i, 0 \leq i \leq n$, where $c(\neq 0) \in GF(s)$. The total number of points in a $PG(n, s)$ is therefore

$$P_n = \frac{s^{n+1}-1}{s-1}. \tag{A.4.1}$$

A t-flat in a $PG(n, s)$ consists of points whose coordinates satisfy a set of $(n-t)$ linearly independent homogeneous equations

$$\begin{aligned}
a_{10}x_0 + a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\
&\vdots \\
a_{(n-t)0}x_0 + a_{(n-t)1}x_1 + \cdots + a_{(n-t)n}x_n &= 0,
\end{aligned}$$

or, $A\boldsymbol{x} = \mathbf{0}$, where $A = (a_{ij})$, $1 \leq i \leq n-t, 0 \leq j \leq n$ and $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)'$. Since the rank of A is $n-t$, the equation $A\boldsymbol{x} = \mathbf{0}$ has $t+1$ linearly independent solutions. Any linear combination of these linearly independent solutions with combining coefficients from $GF(s)$ is also a solution. Hence the number of solutions is s^{t+1} and the number of distinct points lying on a t-flat is

$$P_t = \frac{s^{t+1}-1}{s-1}. \tag{A.4.2}$$

It can be seen that the number of distinct t-flats in $PG(n, s)$, denoted by $\phi(n, t, s)$, is given by

$$\begin{aligned}
\phi(n,t,s) &= \frac{P_n(P_n-P_0)(P_n-P_1)\cdots(P_n-P_{t-1})}{P_t(P_t-P_0)(P_t-P_1)\cdots(P_t-P_{t-1})} \\
&= \frac{(s^{n+1}-1)(s^n-1)\cdots(s^{n-t+1}-1)}{(s^{t+1}-1)(s^t-1)\cdots(s-1)}.
\end{aligned} \tag{A.4.3}$$

It can also be shown that the total number of of distinct t-flats through a fixed point is

$$\phi(n-1, t-1, s), \tag{A.4.4}$$

and the number of distinct t-flats through a pair of fixed points is

$$\phi(n-2, t-2, s). \tag{A.4.5}$$

From (A.4.3), it is also seen that

$$\phi(n, t, s) = \phi(n, n-t-1, s). \tag{A.4.6}$$

We consider next finite Euclidean geometries. As before, let s be a prime or a prime power. An ordered set of n elements $(x_1, x_2, \ldots, x_n)$, where the x_i's are elements of $GF(s)$, is called a point in a finite Euclidean geometry, $EG(n, s)$. A pair of points $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are the same if and only if $x_i = y_i$ for $1 \leq i \leq n$. Thus, the total number of points in $EG(n, s)$ is

$$E_n = s^n. \tag{A.4.7}$$

All points satisfying a set of $(n-t)$ consistent and linearly independent equations

$$\begin{aligned} a_{10} + a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{(n-t)0} + a_{(n-t)1}x_1 + \cdots + a_{(n-t)n}x_n &= 0 \end{aligned}$$

are said to form a t-flat in $EG(n, s)$. The total number of points on a t-flat is given by

$$E_t = s^t. \tag{A.4.8}$$

The number of distinct t-flats is

$$s^{n-t}\phi(n-1, t-1, s) = \phi(n, t, s) - \phi(n-1, t, s). \tag{A.4.9}$$

The total number of distinct t-flats passing through a fixed point is

$$\phi(n-1, t-1, s) \tag{A.4.10}$$

and that through a pair of points is

$$\phi(n-2, t-2, s). \tag{A.4.11}$$

Consider all points of $PG(n,s)$ for which $x_0 = 0$. Clearly, this is an $(n-1)$-flat in $PG(n,s)$. Such a flat is called an $(n-1)$-flat at infinity and all points lying on this flat are called points at infinity. The remaining points are called finite points. Since the finite points of a $PG(n,s)$ have their first coordinate $x_0 \neq 0$, a typical finite point, say $(x_0, x_1, \ldots, x_n)$ can be written as $(1, x'_1, x'_2, \ldots, x'_n)$ where for $1 \leq i \leq n$, $x'_i = x_i/x_0$. One can thus establish a 1-1 correspondence between a finite point of the type $(1, x'_1, \ldots, x'_n)$ of a $PG(n,s)$ and a point $(x'_1, x'_2, \ldots, x'_n)$ of an $EG(n,s)$. A t-flat of $PG(n,s)$ is said to be wholly at infinity if all the points on this flat are at infinity. All other flats are called finite. To any finite t-flat of $PG(n,s)$ given by the equations $a_{i0}x_0 + a_{i1}x_1 + \cdots + a_{in}x_n = 0$, $1 \leq i \leq n-t$, let there correspond a t-flat of $EG(n,s)$ given by the equations $a_{i0} + a_{i1}x_1 \cdots + a_{in}x_n = 0$, $1 \leq i \leq n-t$. The latter set of equations are consistent if the t-flat of $PG(n,s)$ is finite. Thus, there is a 1-1 correspondence between t-flats of $EG(n,s)$ and finite t-flats of $PG(n,s)$. From the above discussion, it is seen that an $EG(n,s)$ can be obtained from a $PG(n,s)$ by omitting all points at infinity and the t-flats wholly lying at infinity.

Conversely, from an $EG(n,s)$, one can construct a $PG(n,s)$ by considering the points in $EG(n,s)$ as the finite points of $PG(n,s)$ and adding the $(n-1)$-flat at infinity with $x_0 = 0$ along with the distinct points lying on this flat.

As an illustration, consider a finite projective geometry $PG(n,s)$ with $n = 2$ and $t = 1$. This is known as a finite projective plane of order s. The number of points in a finite projective plane, by (A.4.1), is $s^2 + s + 1$. The number of 1-flats, which are called lines, by (A.4.3) is

$$\phi(2,1,s) = \frac{(s^3-1)(s^2-1)}{(s^2-1)(s-1)} = s^2 + s + 1.$$

Through any point, there are $s+1$ lines and on any given line, there are $s+1$ distinct points. Through a given pair of points, there is exactly one line. Cutting out a line (the line at infinity) and all points lying on it, one gets an affine plane or, $EG(2,s)$ with $t = 1$, which has s^2 points and $s^2 + s$ lines. Through any point there are $s+1$ lines and on each line, there are s distinct points. Also, through a given pair of lines, there is precisely one line.

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INCOMPLETE BLOCK DESIGNS

This book presents a systematic, rigorous and comprehensive account of the theory and applications of incomplete block designs. All major aspects of incomplete block designs are considered by consolidating vast amounts of material from the literature including the classical incomplete block designs, like the balanced incomplete block (BIB) and partially balanced incomplete block (PBIB) designs. Other developments like efficiency-balanced designs, nested designs, robust designs, C-designs and alpha designs are also discussed, along with more recent developments in incomplete block designs for special types of experiments, like biological assays, test-control experiments and diallel crosses, which are generally not covered in existing books. Results on the optimality aspects of various incomplete block designs are reviewed in a separate chapter that also includes recent optimality results for test-control comparisons, parallel-line assays and diallel cross experiments.

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