BIRTH & DEATH PROCESS

Any model takes into account the essentials of a phenomenon. It may be expressing the phenomenon in symbols or logical, involving mathematical relationships or statement.

It may be changing with time and provides an approximation to the real world situation.

Newton's laws of motion, laws of thermodynamics etc are examples of mathematical model which are useful in engineering business or industry but is does not work in real life.

In the real life, improvements can be achieved by introducing random variables or chance factors in the model. One can not predict the date of death of an individual, but one can predict the chance of death before the next birthday.

Life insurance companies calculate the chance of death and use it to calculate the premium amount for different categories of persons in different age groups, in such situations introducing chance factors or random variables in the model, will improve it, making it closer to reality, the model containing chance factor is called Stochastic model

Stochastic Process

↓

Marcov Process

↓

Birth Death Process

MARCOV PROCESS :

If $\{X(t) , t \in T\}$ which is a Stochastic process such that the given value X, the value of X(t) t>s does not depend on the value of X(u) u>s then such process is said to be Marcov process.

BIRTH DEATH PROCESS:

An important class of Marcov process is called birth death process.

Its state space is countable , taken to be without loss of generality.

For example , let X(t) denote the population size at time t, it can be increased by birth and decreased by death.

The birth and death rates are may depend on time t, we have time homogeneous transition probability.

Birth Process:
Introduction:
If one allows the chance of an event , occurring at a given instant of time to depend upon the number of events that have already occurred in the study of a population growth. Birth may be interrupted as an event whose prob. is depending upon the no. of parents. Here the event may refer to the birth of an individual.

Assumption of the birth process:

In the classified Poisson process, we assume that the conditional prob. is constant. Here the prob. that k events occur between t and t+h, given that n events occurred by epoch t is given by

$$P_k(h) = P\{N(h) = k / N(t) = n\}$$

$$= \lambda h + O(h); k = 1$$

$$= O(h); k >= 2$$

$$= 1 - \lambda h + O(h); k = 0$$

$P_k(h)$ is independent of n as well as t .We can generalize the process by considering that λ is not a constant but is a function of n or t or both, the resulting process will still be Marcovian in character.

Here we consider that λ is a function of n, the population size at the instant we assume that

$$P_k(h) = P\{N(h) = k / N(t) = n\}$$

$$= \lambda_n h + O(h); k = 1$$

$$= O(h); k >= 2$$

$$= 1 - \lambda_n h + O(h); k = 0$$

We shall have the following equation corresponding to Poisson process.

$$P_n(t+h) = P_n(1-\lambda_n h) + P_{n-1}(t)\lambda_{n-1}h + O(h)$$

$$P_n(t+h) - P_n = -\lambda_n h P_n + P_{n-1}(t)\lambda_{n-1}h + O(h)$$

$$\frac{P_n(t+h) - P_n}{h} = -\frac{\lambda_n h P_n(t)}{h} + \frac{P_{n-1}(t)\lambda_{n-1}h}{h} + \frac{O(h)}{h}$$

$$\lim h \to 0 \frac{P_n(t+h) - P_n}{h} = \lim h \to 0 \frac{\lambda_n h P_n(t)}{h} + \lim h \to 0 \frac{P_{n-1}(t)\lambda_{n-1}h}{h} + \lim h \to 0 \frac{O(h)}{h}$$

$$P_n^1(t) = -\lambda_n P_n(t) + P_{n-1}(t)\lambda_{n-1} \quad \text{..............................1}$$

For n=0

$$P_0(t+h)-P_0 = -\lambda_{n0} h P_0 + O(h)/h$$

$$\lim h \to 0 \frac{P_0(t+h)-P_0}{h} = \lim h \to 0 \frac{\lambda_0 h P_0(t)}{h} + \lim h \to 0 \frac{O(h)}{h}$$

$$P_0^1(t) = -\lambda_0 P_0 \quad \text{..2}$$

The system of equation 1 and 2 is called system of birth process.

This system of equation is to be solved with initial condition for obtaining distribution function of birth process.

Here dist. Function will be obtained under Yule-Fury Process hence the pure birth process is called Yule-Fury process.

Yule-Fury Process:
Consider a population whose members are either physical or biological entities. Suppose that members can give birth to new members but can not die. We assume that in an interval of length h using each member has $\lambda h + O(h)$ or giving birth to a new member.

If N(t) denote the total no. of members by epoch t and $P_n(t) = P\{N(t) = n\}$ then by putting $\lambda_n = n\lambda$ in equation 1 and 2 be obtained a system of birth process which is called Yule-Fury birth process.

Probability mass function of Yule-Fury birth process:

From 1

$$P_n^1(t) = -n\lambda\ P_n(t) + P_{n-1}(t)\lambda\ (n-1)$$

$$P_0^1(t) = -(0)\ \lambda\ P_0(t)$$

=0

$P_0(t) = 0$ ………………………………………4

Suppose that the initial condition is given by t=0, n=1, $P_1(0) = 1$

Then $P_i(t) = 0; i > 2$

From 3 n=1;

$P_1^{\ 1}(t) = -\lambda\ P_1(t)$ +0

$$\frac{\partial P_1(t)}{P_1(t)} = -\lambda \int \partial t$$

$\log P_1(t) = -\lambda t + C$

$P_1(t) = \ell^{-\lambda t + C}$

By putting t=0 then $P_1(0) = \ell^{-\lambda(0)} \ell^{c}$

1= ℓ^{c}

$P_1(t) = \ell^{-\lambda t}$

n=2

$P_2^1(t) = -2\lambda \; P_2(t) + \lambda \; P_1(t)$

$\frac{\partial}{\partial t} P_2(t) + \; 2\lambda P_2(t) = \lambda \; \ell^{-\lambda t}$ ……………………………5

Multiply both sides by $\ell^{2\lambda t}$

$$\ell^{2\lambda t}\frac{\partial}{\partial t}P_2(t)+\ell^{2\lambda t}\,2\lambda P_2(t)=\lambda\ell^{-\lambda t}\;\ell^{2\lambda t}$$

$$\frac{\partial}{\partial t}\left[\ell^{2\lambda t}P_2(t)\right]=\lambda\ell^{\lambda t}$$

$$\int\partial P_2(t)\ell^{2\lambda t}=\lambda\int\ell^{\lambda t}dt$$

$$P_2(t)\,\ell^{2\lambda t}=\lambda\ell^{\lambda t}/\lambda+C_1$$

Under boundary condition $P_2(0)=0,$

We have $P_2(0)\;\ell^{2\lambda(0)}=\ell^{\lambda(0)}+C_1$

0=1+C_1

C_1=-1

Hence $P_2(t)\ell^{2\lambda t} = \ell^{\lambda t}$

$$P_2(t) = \ell^{-\lambda t} - \ell^{-2\lambda t}$$

$$= \ell^{-\lambda t}\left[1-\ell^{-\lambda t}\right]$$

$$P_1(t) = \ell^{-\lambda t}\left[1-\ell^{-\lambda t}\right]^{1-1}$$

$$P_2(t) = \ell^{-\lambda t}\left[1-\ell^{-\lambda t}\right]^{2-1}$$

Similarly we can get

$$P_n(t) = \ell^{-\lambda t}\left[1-\ell^{-\lambda t}\right]^{n-1}$$

Which is the p.m.f. of Yule – Fury birth process, which is the form of geometric distribution.

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Show that total prob. of pure birth process is 1.

P(S,t)= $\ell^{-\lambda t}$ S $\left[\dfrac{1}{1-(1-\ell^{-\lambda t})}\right]$

Let A=$\ell^{-\lambda t}$ and B=$\left[1-\ell^{-\lambda t}\right]^{1}$

P(S,t)=$\dfrac{AS}{1-BS}$ = $AS(1-BS)^{-1}$

$= AS\left[1+BS+B^2S^2+.........\right]$

$= A\left[S+BS^2+B^2S^3+.........\right]$

P(S,t)=$A\sum_{x=1}^{\infty} B^{x-1}S^{x}$ ……………………………1

We know that

P(S,t)= $\sum P_x(\lambda) S^x$

$$P_x(t) = AB^{x-1}$$

$$P_x(t) = \ell^{-\lambda t} \left[1 - \ell^{-\lambda t}\right]^{x-1}$$

$$\sum P_x(t) = \sum \ell^{-\lambda t} \left[1 - \ell^{-\lambda t}\right]^{x-1}$$

$$= \ell^{-\lambda t} \left[1 + (1 - \ell^{-\lambda t}) + (1 - \ell^{-\lambda t})^2 + \ldots\ldots\ldots\ \ldots\right]$$

$$= \ell^{-\lambda t} \left[1 - (1 - \ell^{-\lambda t})\right]^{-1}$$

$$= \ell^{-\lambda t} \left[\ell^{-\lambda t}\right]^{-1}$$

$$=1$$

Mean of Pure birth process:

Pure birth process follows geometric distribution and mean of geometric dist. Is given by $\ell^{\lambda t}$ so E(x,t)= $\ell^{\lambda t}$

E(x,t)= $\sum XP\,(x,t)$

$$= \sum X\,\ell^{-\lambda t}\left[1-\ell^{-\lambda t}\right]^{x-1}$$

$$= \ell^{-\lambda t}\left[1+2(1-\ell^{-\lambda t})+3(1-\ell^{-\lambda t})^{2}+..........\ ..\right]$$

$$= \ell^{-\lambda t}\left[1-(1-\ell^{-\lambda t})^{-2}\right]$$

$$= \ell^{-\lambda t}\,\ell^{2\lambda t}$$

$= \ell^{\lambda t}$ =Mean

Variance of pure birth process is given by $\ell^{\lambda t}(\ell^{\lambda t} - 1)$

PURE DEATH PROCESS:

Introduction:

The pure death process or simple death process is exactly analogous to pure birth process except that in a pure death process X(t) is decreased rather than increasing by the occurrence of an event.

ASSUMPTIONS:

1. At the time zero the system is in state x_0 i.e. $x_0 = x_0$ >=1(size of the population)
2. If at time t the system is in estate x(x=1,2,3….) then the prob. of transition from $x \rightarrow$ x -1in the interval (t,t+h) is $\lambda h + O(h)$
3. The prob. of transition from the state $x \rightarrow x_{i-1} (i > 1)$ is $O(h)$
4. The prob. of no change is $1 - \lambda h + O(h)$

$$P_n(t+h)=P_n(t)\{1-\lambda_n h+O(h)\}+P_{n+1}(t)\{\lambda_{n+1}h+O(h)\}+O(h)$$

$$\lim h \to 0 \frac{P_n(t+h)-P_n(t)}{h}=\frac{d}{dt}P_n(t)=-\lambda_n P_n(t)+P_{n+1}(t)\lambda_{n+1}$$

further let us consider that the death is linear.

i.e. $\lambda_n = n\lambda$, λ >=0, n>=1

with this assumption , we get the different equations for the simple death process as

$$\frac{d}{dt}P_n(t)=-\lambda n P_n(t)+\lambda(n+1)P_{n+1}(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1$$

is to be solved with the initial conditions

$$P_x(0) = S^x, x = 1..if..x = x_0$$
$$x = o \quad ;O.W$$

Using the method of generating function is obtained the expression for $P_x(t)$, put

$$F(S,t) = \sum_{x=0}^{x_0} S^x P_x(t)$$

Multiplying the equ. 1 both side by S^x we get

$$\frac{d}{dt}P_n(t)\ S^x = -\lambda x P_n(t) S^x + \lambda(n+1)P_{n+1}(t)S^x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2$$

Taking summation over x of both side of 2, we have

$$\sum \frac{d}{dt}P_n(t)\, S^x = -\lambda \sum xP_x(t)S^x + \lambda \sum (x+1)P_{n+1}(t)S^x$$

$$\sum \frac{d}{dt}P_n(t)\, S^x = -\lambda S \sum xP_x(t)S^{x-1} + \lambda \sum (x+1)P_{n+1}(t)S^{x+1-1} \quad \ldots\ldots\ldots\ldots 3$$

We also know that P.G.F. can be defined as

$$F(S,t) = \sum_{x=0}^{x_0} S^x P_x(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 4$$

Using 3 and 4 we have

$$\frac{\partial}{\partial t}F(S,t) = -\lambda S\frac{\partial}{\partial S}F(S,t) + \lambda\frac{\partial}{\partial S}F(S,t)$$

$$= \frac{\partial}{\partial S}F(S,t)\ \lambda(1-S)$$

$$\frac{\partial}{\partial t}F(S,t) = \ -\frac{\partial}{\partial S}F(S,t)\ \lambda(S-1)$$

$$\frac{\partial}{\partial t}F(S,t) + \frac{\partial}{\partial S}F(S,t)\ \lambda(S-1) \ =0$$

Its subsidiary solution will be

$$\frac{\partial t}{1} = \frac{\partial s}{\lambda(S-1)} = \frac{\partial F(S,t)}{0} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.5$$

Using 1 and 3

$$\frac{\partial t}{1} = \frac{\partial F(S,t)}{0}$$

$$\int \partial F(S,t) = \int 0$$

$$F(S,t) = Constant$$

Using 1 and 2

$$\frac{\partial t}{1} = \frac{\partial s}{\lambda(S-1)}$$

$$\int \partial t = \int \frac{\partial s}{\lambda(S-1)}$$

$$\lambda t = \log(S-1) + \log C$$

$$\lambda t = \log(S-1)C$$

$$(S-1)C = \ell^{\lambda t}$$

$(S-1)C = \ell^{\mu t}$ where $\lambda = \mu$ =death constant

$(S-1)\ \ell^{-\mu t}$ =1/C= C^1

$F(S,t)$ =Constant

The general solution is now

$F(S,t)$ =f(S-1)$\ell^{-\mu t}$ }…………………………………6

Let f($Z_1) = (1+Z_1)^{X_0}$

For Z=S-1

F($Z_1) = (1+S-1)^{X_0}$

$= S^{X_0}$

hence we see a function f (Z_1) such that above result holds, we observe that

f $(Z_1) = (1+Z\ )^{X_0}$ satisfies the condition

$F(S,t)$ =f{(S-1) $\ell^{-\mu t}$ }

= {1+(S-1) $\ell^{-\mu t}$ $\}^{X_0}$

={1+S-1/ $\ell^{\mu t}$ $\}^{X_0}$

$$= \left\{ \frac{\ell^{\mu t} + S - 1}{\ell^{\mu t}} \right\}^{X_0}$$

$$F(S,t) = \left(\ell^{-\mu t}\right)^{X_0} \left(\ell^{\mu t} + S - 1\right)^{X_0}$$

$$= \left(\ell^{-\mu t}\right)^{X_0} \left[\left(\ell^{\mu t} - 1\right)^{X_0} \left\{ 1 + \frac{S}{\ell^{\mu t} - 1} \right\}^{X_0} \right]$$

$$= \left(\ell^{-\mu t}\right)^{X_0} \left(\ell^{\mu t} - 1\right)^{X_0} \left\{ 1 + S\left(\ell^{\mu t} - 1\right)^{-1} \right\}^{X_0}$$

$$= \left(\ell^{-\mu t}\right)^{X_0} \left(\ell^{\mu t} - 1\right)^{X_0} \left[1 + \left\{ \ell^{-\mu t} \left(1 - \ell^{-\mu t}\right)^{-1} \right\} S \right]^{X_0}$$

$$=(\ell^{-\mu t})^{X_0}\ (\ell^{\mu t}-1)^{X_0}\ \left[1+{}^{x_0}C_1\left[\left\{\ell^{-\mu t}(1-\ell^{-\mu t})^{-1}\right\}s\right]+{}^{x_0}C_2\left[\left\{\ell^{-2\mu t}(1-\ell^{-\mu t})^{-2}\right\}s^2\right]+\ldots\ldots\right]$$

$$=(\ell^{-\mu t})^{X_0}\ (\ell^{\mu t}-1)^{X_0}\ (\ell^{\mu t})^{X_0}\ \left[1+{}^{x_0}C_1\left[\left\{\ell^{-\mu t}(1-\ell^{-\mu t})^{-1}\right\}s\right]+\ldots+{}^{x_0}C_x\left[\left\{\ell^{-x\mu t}(1-\ell^{-\mu t})^{-x}\right\}s^x\right]\right]$$

The coefficient of s^x in the expansion of p.g.f. will give the p.m.f. of x .

$$P(X)=(\ell^{-\mu t}\ell^{\mu t})^{X_0}\ (1-\ell^{-\mu t})^{x_0}\ {}^{X_C}Cx\ (\ell^{-\mu t})^{X_0}\ (1-\ell^{-\mu t})^{-x}$$

$$=\ {}^{X_C}Cx\ (\ell^{-\mu t})^{X_0}\ (1-\ell^{-\mu t})^{x_0-x}$$

If p= $\ell^{-\mu t}$,q=1-p

So this P(x) is the p.m.f. of a binomial dist.

Mean of Death process=np= $x_0 \ \ell^{-\mu t}$

Variance of death process=npq= $x_0 \ \ell^{-\mu t}$ (1-$\ell^{-\mu t}$)

Here dist. is binomial then total prob. is also one.

BIRTH AND DEATH PROCESS:
ASSUMPTION:

1. If at a time t , the system in state x(x=1,2…) then prob. of transition from x$\rightarrow x+1$ in (t,t+h) is $\lambda_x h + O(h)$
2. If at a time t , the system in state x(x=1,2…) then prob. of transition from x$\rightarrow x-1$ in (t,t+h) is $\mu_x h + O(h)$
3. The prob. of transition to a state other then a neighboring state is ($x-1$ or x+1) O(h).
4. The prob. of no change is 1-($\lambda_x + \mu_x$)h+O(h)
5. The state x=0 is in absorbing state.

These assumptions lead to the equation,

$$P_x(t+h) = P_{x-1}(\lambda_{x-1}h + O(h)) + P_x(t)[1-(\mu_x + \lambda_x)h + O(h)] + P_{x+1}(t)[\mu_{x+1}h + O(h)] + O(h)$$

$$\lim h \to 0 \frac{P_x(t+h) - P_x(t)}{h} = \frac{h\left[P_{x-1}\lambda_{x-1} - P_x(t)(\mu_x + \lambda_x) + P_{x+1}(t)\mu_{x+1}]\right]}{h} +$$

$$\lim h \to 0 \frac{O(h)}{h}$$

$$\frac{\partial}{\partial t} P_x(t) = P_{x-1}\lambda_{x-1} - P_x(t)(\mu_x + \lambda_x) + P_{x+1}(t)\mu_{x+1}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1$$

Now for x=0.

$$P_0(t+h) = P_0(t)[1-(\mu_0 + \lambda_0)h + P_1(t)[\mu_1 h + O(h)]$$

$$\lim h \to 0 \frac{P_0(t+h) - P_0(t)}{h} = \lim h \to 0 \frac{h\left[P_1(t)\mu_1 - P_0(t)(\mu_0 + \lambda_0)\right]}{h} + \lim h \to 0 \frac{O(h)}{h}$$

$$\frac{\partial}{\partial t} P_0(t) = P_1(t).\mu_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 2$$

as $\lambda_0 = \mu_0 = 0$, and $\mu_1 = \mu$

Let us consider the case of linear birth death process.

If at time zero, the system is in state $x = x_0$ $(0 < x < \infty)$

the initial conditions are $P_x(0) = S_x$

$x_0 = 1$ if $x = x_0$

$= 0$, otherwise

This represents a birth death process, the coefficients of λ_x and μ_x are arbitrary functions of birth death equations

i.e.

$\mu_x = \mu x$ and $\lambda_x = \lambda x$

Define the p.g.f. by

F(S,t)= $\sum S^x P_x(t)$ ……………………A

So the linear birth death process,

From 1 and 2 we get

$$\frac{\partial}{\partial t} P_x(t) = P_{x-1}\lambda (x-1) - P_x(t)(\mu + \lambda)\mathrm{x} + \mathrm{P}_{\mathrm{x}+1}(t)\mu (x+1)]$$ ………………..3

$$\frac{\partial}{\partial t} P_0(t) = \mu P_1(t)$$ …………………………………….4

Multiplying both sides of 3 by S^x and taking summation over entire range of X, we get

$$\sum_{x=0}^{\infty}\frac{d}{dt}S^x\ P_x(t)=\sum_{x=0}^{\infty}S^x\lambda(x-1)P_{x-1}(t)-\sum_{x=0}^{\infty}S^x x(\lambda+\mu)P_x(t)+\sum_{x=0}^{\infty}S^x\mu(x+1)P_{x+1}(t)\ \ldots\ldots 5$$

From A we can have

$\frac{d}{dt}$ F(S,t)=

$$\sum_{x=0}^{\infty}xS^{x-1}P_x(t)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{B}$$

5 can be rewritten as

$$\sum_{x=0}^{\infty}\frac{d}{dt}S^x\ P_x(t)=S^2\lambda\sum_{x=0}^{\infty}S^{x-2}(x-1)P_{x-1}(t)-S(\lambda+\mu)\sum_{x=0}^{\infty}S^{x-1}x(\lambda+\mu)P_x(t)+\sum_{x=0}^{\infty}S^x\mu(x+1)P_x$$

Using A and B we get

$\frac{d}{dt}$ $S^2\lambda \frac{d}{dt}$ F(S,t)-S$(\lambda+\mu)\frac{d}{dt}$ F(S,t)+$\mu\frac{d}{dt}$ F(S,t)

=$\frac{d}{dt}$ F(S,t)[$S^2\lambda$ -S$(\lambda+\mu)$ +μ]

=($\lambda S - \mu$)(S-1)$\frac{d}{dt}$ F(S,t)

Now complementary solution is

$$\frac{dt}{1} = \frac{ds}{-(\lambda S-\mu)(S-1)} = \frac{dF(s,t)}{0}$$

Using first and third term

$$0\int dt = \int dF(s,t)$$

i.e F(S,t)=constant.

Using first and second term

$$\frac{dt}{1} = \frac{ds}{-(\lambda S-\mu)(S-1)} = \frac{-1}{(\mu-\lambda)}\left[\frac{\lambda}{\lambda S-\mu} - \frac{1}{S-1}\right]ds$$

$$(\lambda-\mu)\int dt = \log(\lambda S-\mu) - \log(S-1) + \log C$$

$$(\lambda-\mu)\,\mathrm{t} = \log\frac{\lambda S-\mu}{S-1} + \log \mathrm{C}$$

$$\ell^{(\lambda-\mu)t} = \frac{\lambda S-\mu}{S-1}\,\mathrm{C}$$

$$\frac{\mu-\lambda S}{1-S}\,\frac{1}{\ell^{(\lambda-\mu)t}} = \frac{1}{C}$$

$$\frac{\mu-\lambda S}{1-S}\,\ell^{-(\lambda-\mu)t} = C^1$$

So the general solution is

F(S,t)=f[$\left\{\frac{\mu-\lambda S}{1-S}\right\}\ell^{-(\lambda-\mu)t}$]

=f(z) say

Where f(0) is any arbitrary function

Let us assume that x= x_0 =1 , then

F(S,0)= $\sum_{x=0}^{\infty} S^1 P_x(0)$

$= S\sum_{x=0}^{\infty} P_x(0)$

=S(1)

=S

so, $S=F(S,0)=F\{\frac{\mu-\lambda S}{1-S}\}$ at $t=0$

Let $F(z_1)=\frac{\mu-z}{\lambda-z}$

$$F(s,t)=\frac{\mu-\left(\frac{\mu-\lambda s}{1-s}\right)\ell^{-(\lambda-\mu)t}}{\lambda-\left(\frac{\mu-\lambda s}{1-s}\right)\ell^{-(\lambda-\mu)t}}$$

$$= \frac{(1-s)(\ell^{(\lambda-\mu)t}\mu) - (\mu - \lambda s)}{\lambda(1-s)(\ell^{(\lambda-\mu)t}) - (\mu - \lambda s)}$$

$$= \frac{\mu[(\ell^{(\lambda-\mu)t}) - 1] + s(\lambda - \mu\ell^{(\lambda-\mu)t})}{[\lambda(\ell^{(\lambda-\mu)t}) - \mu] + s\lambda[1 - \ell^{(\lambda-\mu)t}]}$$

$$= \frac{\mu[(\ell^{(\lambda-\mu)t}) - 1][1 + \left\{\dfrac{\lambda - \mu(\ell^{(\lambda-\mu)t})}{\mu\{(\ell^{(\lambda-\mu)t}) - 1\}}\right\}s]}{[\lambda(\ell^{(\lambda-\mu)t}) - \mu][1 + \left\{\dfrac{\lambda s(1 - \ell^{(\lambda-\mu)t})}{\lambda\ell^{(\lambda-\mu)t} - \mu\}}\right\}]}$$

$$F(s,t) = \frac{\alpha(t)[1 + \left\{\dfrac{\lambda - \mu(\ell^{(\lambda-\mu)t})}{\mu\{(\ell^{(\lambda-\mu)t}) - 1\}}\right\}s]}{1 - \beta(t)s}$$

where $$\alpha(t) = \frac{\mu[(\ell^{(\lambda-\mu)t}) - 1]}{[\lambda(\ell^{(\lambda-\mu)t}) - \mu]}$$

$$\beta(t) = \frac{[(\ell^{(\lambda-\mu)t}) - 1]\lambda}{[\lambda(\ell^{(\lambda-\mu)t}) - \mu]}$$

so $$F(s,t) = \alpha(t)[1 + \left\{\frac{\lambda - \mu(\ell^{(\lambda-\mu)t})}{\mu\{(\ell^{(\lambda-\mu)t}) - 1\}}\right\} s\,][1 - \beta(t)\ s]^{-1}$$

$$= \alpha(t)\,[1 + AS]\ [1 - \beta(t)\ s]^{-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 6$$

where $$A = \left\{\frac{\lambda - \mu(\ell^{(\lambda-\mu)t})}{\mu\{(\ell^{(\lambda-\mu)t}) - 1\}}\right\}$$

$$F(s,t) = \boldsymbol{\alpha(t)}\,[1+AS]\,[1+\boldsymbol{\beta(t)}\,s+\{\boldsymbol{\beta(t)}\,s\}^2+\ldots\ldots]$$

$$= \boldsymbol{\alpha(t)}\,\{\,[A(s+\boldsymbol{\beta(t)}s^2+\boldsymbol{\beta(t)}^2 s^3+\ldots\ldots]$$

$$+[1+\boldsymbol{\beta(t)}\,s+\{\boldsymbol{\beta(t)}\,s\}^2+\ldots\ldots]\}\ldots\ldots\ldots\ldots 7$$

Now collecting the coefficients of s^x in the expansion of 7 ,we get

$$\mathbf{p_0(t) = \alpha(t)} \quad \text{for } x=0$$

$$\mathbf{p_1(t) = \alpha(t)\{A + \beta(t)\}}$$

$$\mathbf{p_2(t) = [A\beta(t) + \beta(t)^2 + \ldots..]\alpha(t)}$$